# Open Quantum System approach to in medium quarkonium

Tom Magorsch

QCD challenges from pp to AA collisions

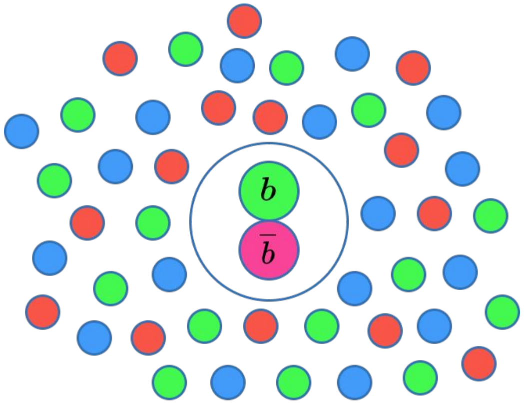
03.09.2024

# Quarkonium suppression

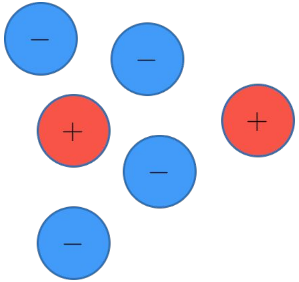
Veres, Gábor I. "Heavy ion physics at CMS and ATLAS: hard probes." *arXiv preprint arXiv:1905.10461* (2019).

Propagation through QGP

$T \approx O(100\mathrm{MeV})$



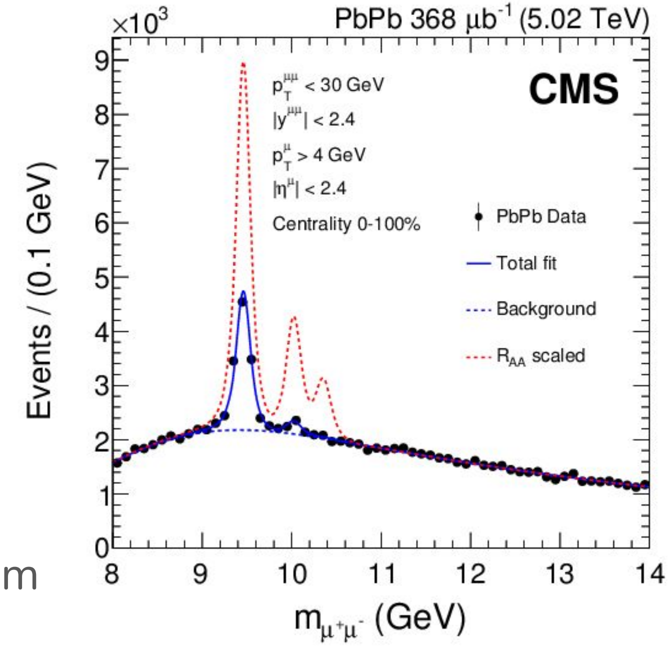

T. Matsui, H. Satz, Phys. Lett. B 178 (1986) 416




$$V(r) = -\frac{\alpha}{r}$$

↓ Debye-screening in medium

$$V(r) = -\frac{\alpha}{r}\mathrm{e}^{-m_D r}$$




- Phenomenological predictions for **bottomonium** from **first principles**

# Open Quantum Systems

- Quantum system not isolated
- Split into System $S$ and Environment $E$

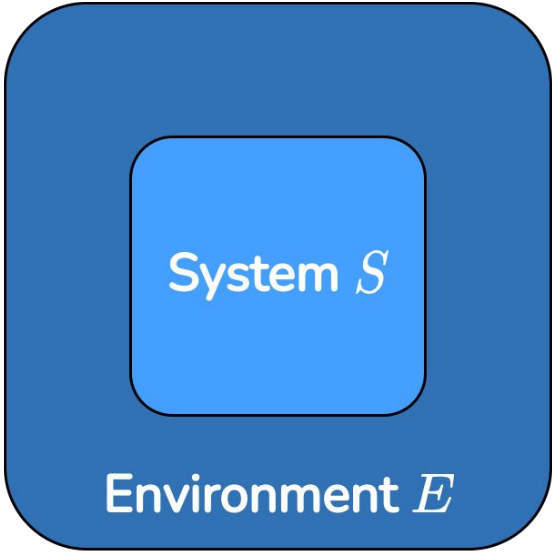
$$H = H_S \otimes I_E + I_S \otimes H_E + H_{\text{int}}$$

- Time evolution by Von-Neumann Equation

$$\frac{d}{dt}\rho = -i[H, \rho]$$

- Not interested in environmental d.o.f.: **Trace out**!

$$\rho_S = \mathrm{Tr}_E[\rho]$$



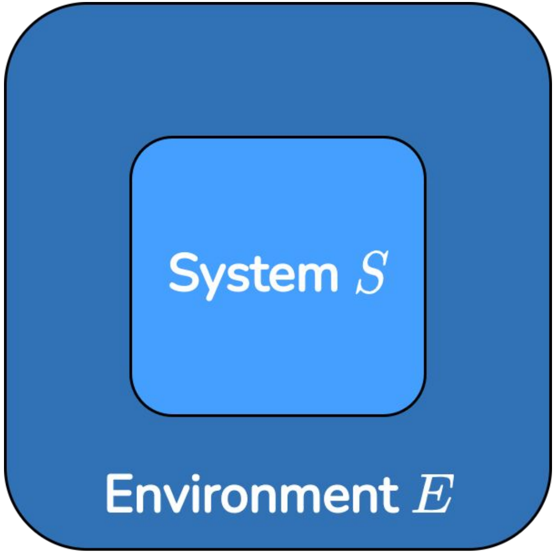
# Open Quantum Systems

- Time evolution by Von-Neumann Equation

$$\frac{d}{dt}\rho = -i[H, \rho]$$

- Environmental d.o.f. not needed Trace out!

$$\rho_S = \mathrm{Tr}_E[\rho]$$




- “Master equation” for the System: **Lindblad Equation**

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \sum_n \left( C_n \rho_S C_n^\dagger - \frac{1}{2}\{C_n^\dagger C_n, \rho_S\} \right)$$

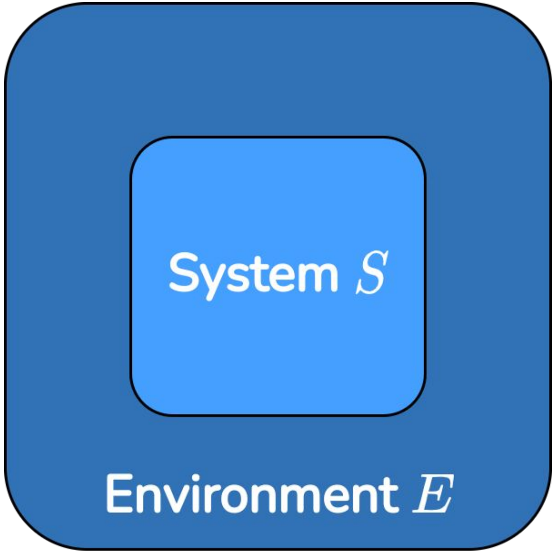
# Open Quantum Systems

- Time evolution by Von-Neumann Equation

$$\frac{d}{dt}\rho = -i[H,\rho]$$

- Environmental d.o.f. not needed Trace out!

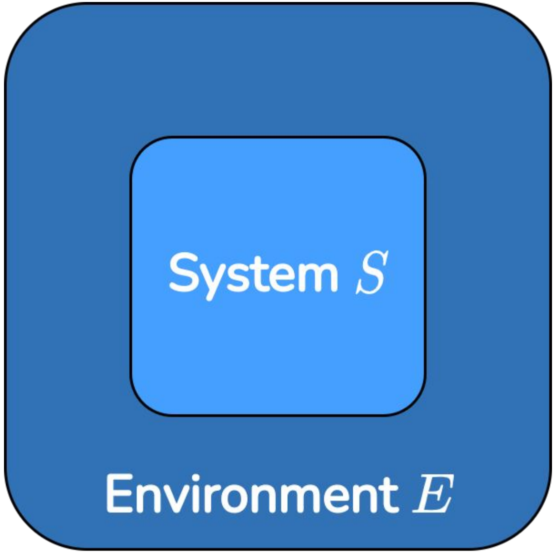
$$\rho_S = \mathrm{Tr}_E[\rho]$$




- “Master equation” for the System: **Lindblad Equation**

$$\frac{d\rho_S}{dt} = -i[H_S,\rho_S] + \sum_n \left( C_n \rho_S C_n^\dagger - \frac{1}{2}\left\{ C_n^\dagger C_n, \rho_S \right\} \right)$$

# Open Quantum Systems




- Time evolution by Von-Neumann Equation

$$\frac{d}{dt}\rho = -i[H, \rho]$$

- Environmental d.o.f. not needed Trace out!

$$\rho_S = \mathrm{Tr}_E[\rho]$$

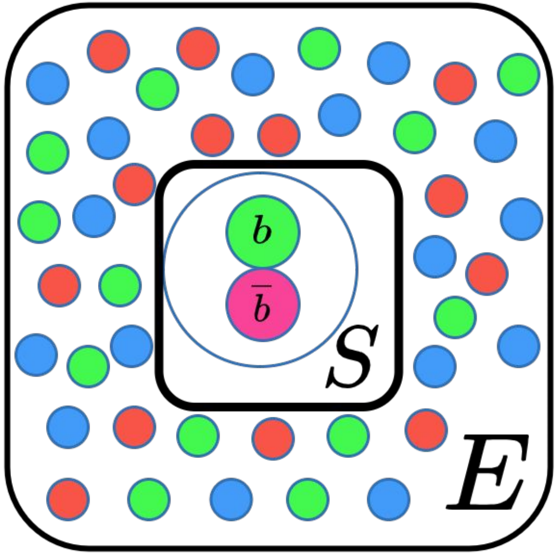
- "Master equation" for the System: **Lindblad Equation**

non-unitary

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \sum_n \left( C_n \rho_S C_n^\dagger - \frac{1}{2}\left\{C_n^\dagger C_n, \rho_S\right\} \right)$$

# OQS for quarkonium

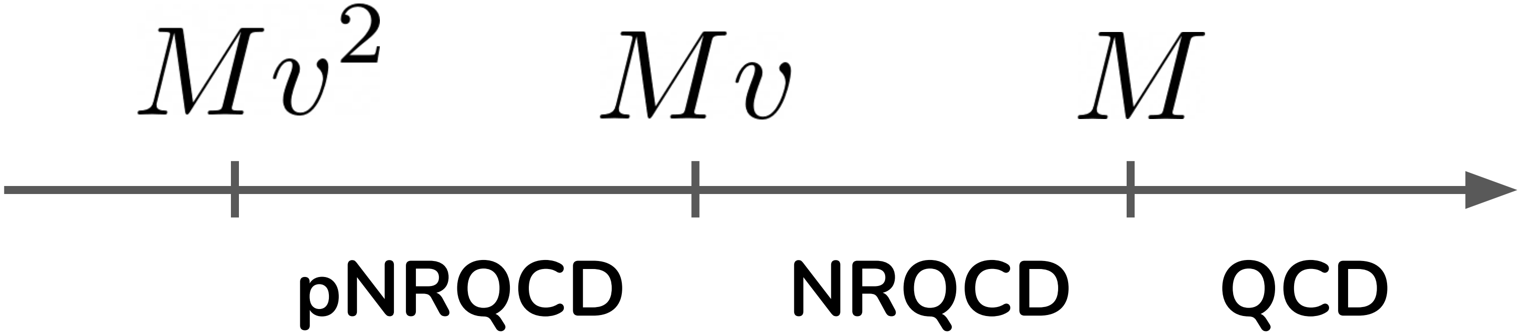
- Quarkonium: System $S$
- QGP: Environment $E$




Aim to describe Quarkonium Suppression by a master equation for encoding the interaction with the QGP

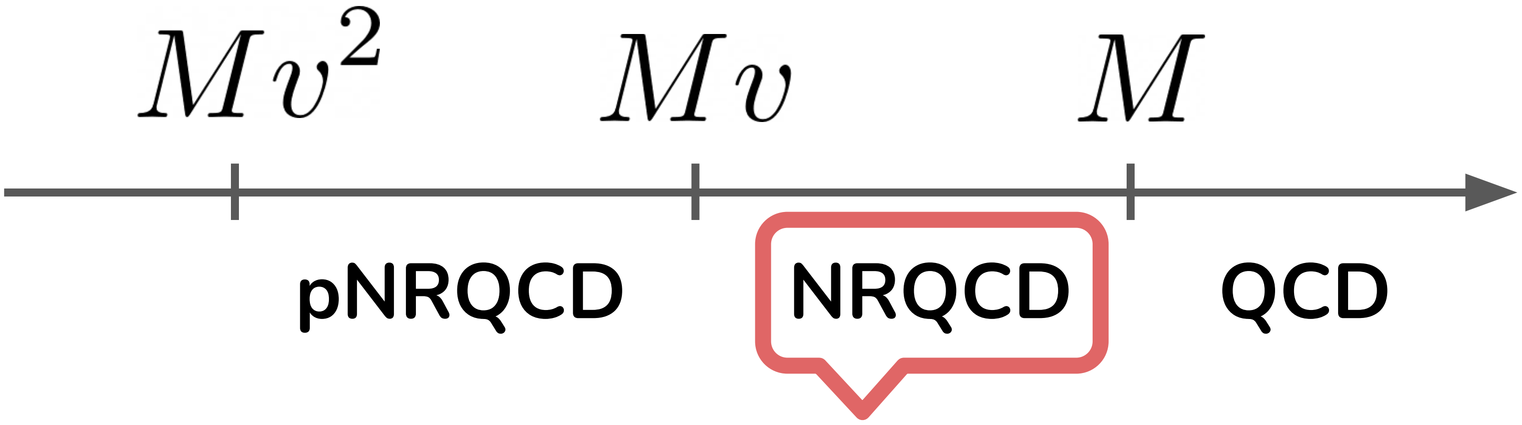
$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \sum_n \left( C_n \rho_S C_n^\dagger - \frac{1}{2}\left\{ C_n^\dagger C_n, \rho_S \right\} \right)$$

# Master equations from EFTs



# Master equations from EFTs

Yukinao Akamatsu, Prog.Part.Nucl.Phys. 123 (2022), 103932
Xiaojun Yao, Int.J.Mod.Phys.A 36 (2021) 20, 2130010




- Degrees of Freedom: Heavy quarks

Lindblad eq. for *Heavy quarks*:

Akamatsu, Y. (2015). *Physical Review D*, *91*(5), 056002.

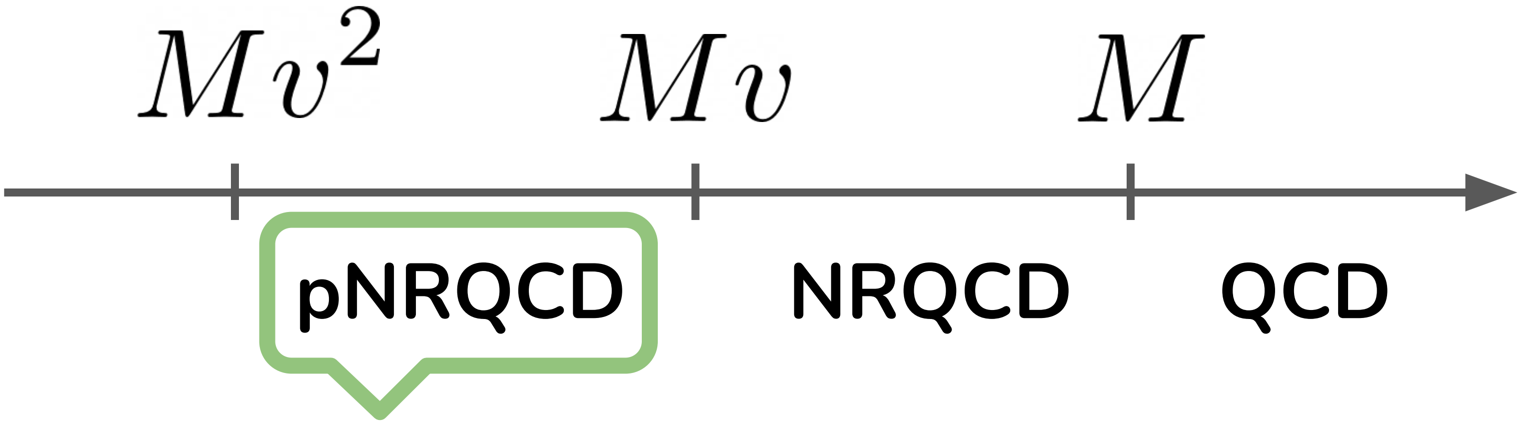
Blaizot, J. P., & Escobedo, M. A. (2018). *Journal of High Energy Physics*, *2018*(6), 1-57.

Lindblad eq. for *Heavy quarkonium*:

Blaizot, J. P., & Escobedo, M. A. (2018). *Physical Review D*, *98*(7), 074007.

Yukinao Akamatsu, Prog.Part.Nucl.Phys. 123 (2022), 103932
Xiaojun Yao, Int.J.Mod.Phys.A 36 (2021) 20, 2130010

# Master equations from EFTs




- Degrees of Freedom: Singlet and octet bound states

Lindblad eq. for *Heavy quarkonium:*

Brambilla, N., Escobedo, M. A., Soto, J., & Vairo, A. (2018). *Physical Review D*, *97*(7), 074009.

Brambilla, N., Escobedo, M. A., Soto, J., & Vairo, A. (2017). *Physical Review D*, *96*(3), 034021.

# pNRQCD master equation

Brambilla, Escobedo, Soto, Vairo: Phys. Rev. D 97 (2018) 7, 074009

$$\frac{d\rho(t)}{dt} = -i\left[H, \rho(t)\right] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2}\left\{ L_i^{m\dagger} L_i^n, \rho(t) \right\} \right)$$

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$L_i^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i, \quad L_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{N_c^2-4}{2(N_c^2-1)} A_i^{oo\dagger} \end{pmatrix}$$

$$L_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_i, \quad L_i^3 = \begin{pmatrix} 0 & \frac{1}{N_c^2-1} A_i^{os\dagger} \\ A_i^{so\dagger} & 0 \end{pmatrix}$$

$$A_i^{uv} = \frac{g^2}{6N_c} \int_0^\infty \mathrm{d}s e^{-ih_u s} r_i e^{ih_v s} \langle E_j^a(s) \Omega(s,0)^{ab} E_j^b(0) \rangle$$

# pNRQCD master equation

Brambilla, Escobedo, Soto, Vairo: Phys. Rev. D 97 (2018) 7, 074009

$$\frac{d\rho(t)}{dt} = -i\,[H,\rho(t)] + \sum_{nm} h_{nm}\left(L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2}\left\{L_i^{m\dagger} L_i^n, \rho(t)\right\}\right)$$

- In general $h_{nm}$ not completely positive:

Master equation *not necessarily* of Lindblad type

$$L_i^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i, \quad L_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{N_c^2-4}{2(N_c^2-1)} A_i^{oo\dagger} \end{pmatrix}$$

$$L_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_i, \quad L_i^3 = \begin{pmatrix} 0 & \frac{1}{N_c^2-1} A_i^{os\dagger} \\ A_i^{so\dagger} & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

# E/T expansion

- Simplify using hierarchy of scales $\pi T \gg E$

$$A_i^{uv} = \frac{g^2}{6N_c}\int_0^\infty \mathrm{d}s e^{-ih_u s} r_i e^{ih_v s}\langle E_j^a(s)\Omega(s,0)^{ab}E_j^b(0)\rangle$$

- Expand exponentials in $E/(\pi T)$
- At **LO** in $E/(\pi T)$ we get

$$A_i^{uv} = \frac{g^2}{6N_c}\int_0^\infty \mathrm{d}s r_i \langle E_j^a(s)\Omega(s,0)^{ab}E_j^b(0)\rangle$$
$$= \frac{r_i}{2}(\kappa - i\gamma)$$

Transport coefficients

Viljamis Talk

# pNRQCD LO Lindblad equation

$$A_i^{uv} = \frac{g^2}{6N_c}\int_0^\infty \mathrm{d}s r_i \langle E_j^a(s)\Omega(s,0)^{ab}E_j^b(0)\rangle$$
$$= \frac{r_i}{2}(\kappa - i\gamma)$$

Transport coefficients

$$L_i^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i, \qquad L_i^1 \propto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i,$$

$$L_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_i, \qquad L_i^3 \propto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_i,$$

Linearly dependent: Lindblad possible!

# pNRQCD LO Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_n \left[ C_i^n \rho(t) C_i^{n\dagger} - \frac{1}{2}\left\{ C_i^{n\dagger} C_i^n, \rho(t) \right\} \right],$$

$$H = \begin{pmatrix} h_s + \frac{r^2}{2}\gamma & 0 \\ 0 & h_o + \frac{N_c^2-2}{2(N_c^2-1)}\frac{r^2}{2}\gamma \end{pmatrix}$$

$$h_{s,o} = \vec{p}^{\,2}/M + V_{s,o}$$

# pNRQCD LO Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_n \left[ C_i^n \rho(t) C_i^{n\dagger} - \frac{1}{2}\left\{ C_i^{n\dagger} C_i^n, \rho(t) \right\} \right],$$

$$H = \begin{pmatrix} h_s + \frac{r^2}{2}\gamma & 0 \\ 0 & h_o + \frac{N_c^2-2}{2(N_c^2-1)}\frac{r^2}{2}\gamma \end{pmatrix}$$

$$h_{s,o} = \vec{p}^{\,2}/M + V_{s,o}$$

Quarkonium Potential

# pNRQCD LO Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_n \left[ C_i^n \rho(t) C_i^{n\dagger} - \frac{1}{2} \left\{ C_i^{n\dagger} C_i^n, \rho(t) \right\} \right],$$

$$H = \begin{pmatrix} h_s + \frac{r^2}{2}\gamma & 0 \\ 0 & h_o + \frac{N_c^2-2}{2(N_c^2-1)} \frac{r^2}{2}\gamma \end{pmatrix}$$

$$C_i^0 = \sqrt{\frac{\kappa}{N_c^2 - 1}} r_i \begin{pmatrix} 0 & 1 \\ \sqrt{N_c^2 - 1} & 0 \end{pmatrix},$$

$$C_i^1 = \sqrt{\frac{\kappa(N_c^2 - 4)}{2(N_c^2 - 1)}} r_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h_{s,o} = \vec{p}^2/M + V_{s,o}$$

# pNRQCD LO Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_n \left[ C_i^n \rho(t) C_i^{n\dagger} - \frac{1}{2}\left\{ C_i^{n\dagger} C_i^n, \rho(t) \right\} \right],$$

$$H = \begin{pmatrix} h_s + \frac{r^2}{2}\gamma & 0 \\ 0 & h_o + \frac{N_c^2-2}{2(N_c^2-1)}\frac{r^2}{2}\gamma \end{pmatrix} \qquad C_i^0 = \sqrt{\frac{\kappa}{N_c^2-1}} r_i \begin{pmatrix} 0 & 1 \\ \sqrt{N_c^2-1} & 0 \end{pmatrix},$$

$$C_i^1 = \sqrt{\frac{\kappa(N_c^2-4)}{2(N_c^2-1)}} r_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$h_{s,o} = \vec{p}^{\,2}/M + V_{s,o}$$

Transport coefficients

# pNRQCD NLO Lindblad equation

$$A_i^{uv} = \frac{g^2}{6N_c}\int_0^\infty \mathrm{d}s e^{-ih_u s} r_i e^{ih_v s} \langle E_j^a(s)\Omega(s,0)^{ab}E_j^b(0)\rangle$$

$$1 + ih_v s$$

$$A_i^{uv} = \frac{r_i}{2}(\kappa - i\gamma) + \kappa\left(-\frac{ip_i}{2MT} + \frac{\Delta V_{uv}}{4T} r_i\right) + \cdots$$

# pNRQCD NLO Lindblad equation

$$A_i^{uv} = \frac{g^2}{6N_c}\int_0^\infty \mathrm{d}s e^{-ih_u s} r_i \underbrace{e^{ih_v s}}_{1 + ih_v s} \langle E_j^a(s)\Omega(s,0)^{ab} E_j^b(0)\rangle$$

$$A_i^{uv} = \frac{r_i}{2}(\kappa - i\gamma) + \kappa\left(-\frac{ip_i}{2MT} + \frac{\Delta V_{uv}}{4T} r_i\right) + \cdots$$

$$L_i^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i, \quad L_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{N_c^2-4}{2(N_c^2-1)} A_i^{oo\dagger} \end{pmatrix}$$

**Not linearly dependent!**

**But corrections of order** $E^2/(\pi T)^2$

# pNRQCD LO Lindblad equation

$$\frac{d\rho(t)}{dt} = -i[H,\rho(t)] + \sum_n \left[ C_i^n \rho(t) C_i^{n\dagger} - \frac{1}{2}\left\{ C_i^{n\dagger} C_i^n, \rho(t) \right\} \right],$$

$$H = \begin{pmatrix} h_s + \frac{r^2}{2}\gamma + \boxed{\frac{\kappa}{4MT}\{r_i, p_i\}} & 0 \\ 0 & h_o + \frac{N_c^2-2}{2(N_c^2-1)}\left(\frac{r^2}{2}\gamma + \boxed{\frac{\kappa}{4MT}\{r_i, p_i\}}\right) \end{pmatrix}$$

$$C_i^0 = \sqrt{\frac{\kappa}{N_c^2-1}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left( r_i + \boxed{\frac{ip_i}{2MT} + \frac{\Delta V_{os}}{4T} r_i} \right) \qquad C_i^1 = \sqrt{\frac{\kappa(N_c^2-4)}{2(N_c^2-1)}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left( r_i + \boxed{\frac{ip_i}{2MT}} \right)$$

$$+ \sqrt{\kappa} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left( r_i + \boxed{\frac{ip_i}{2MT} + \frac{\Delta V_{so}}{4T} r_i} \right),$$

# pNRQCD master equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_n \left[ C^n_{\cancel{i}} \rho(t) C^{n\dagger}_{\cancel{i}} - \frac{1}{2} \left\{ C^{n\dagger}_{\cancel{i}} C^n_{\cancel{i}}, \rho(t) \right\} \right],$$

$$\text{3D} \xrightarrow{\text{Projection on spherical harmonics}} \text{1D + angular momentum}$$

# pNRQCD master equation

$$\frac{d\rho(t)}{dt} = -i[H,\rho(t)] + \sum_n \left[ C^n_{\cancel{i}}\rho(t)C^{n\dagger}_{\cancel{i}} - \frac{1}{2}\left\{ C^{n\dagger}_{\cancel{i}}C^n_{\cancel{i}}, \rho(t) \right\} \right],$$

## Hilbert space:

$$H_c \otimes H_l \otimes H_r$$

**c = s,o**

**l = 0,1,2,...**

**Radial wavefunction**

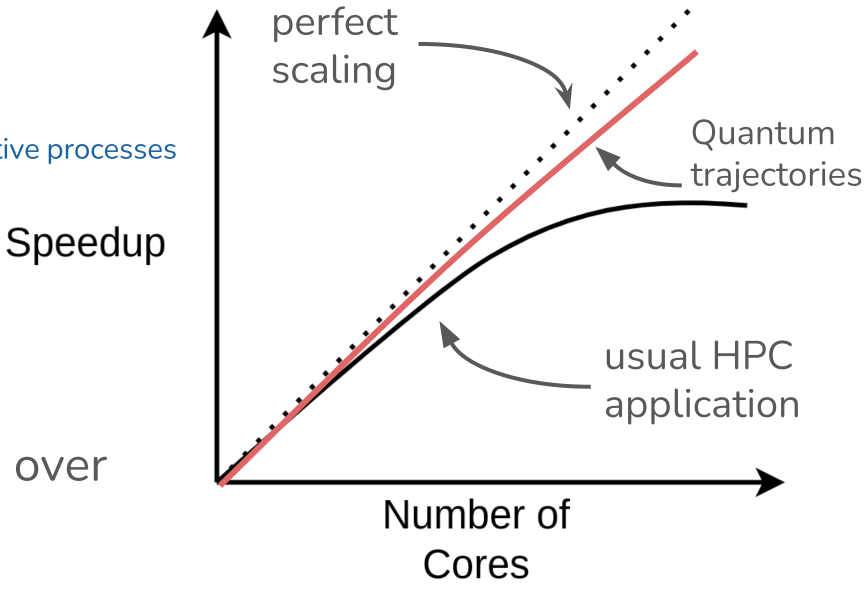
e.g. 2000 point lattice

# Quantum trajectory algorithm

J. Dalibard, Y. Castin, and K. Mølmer, Wave-function approach to dissipative processes in quantum optics, Phys. Rev. Lett. 68 (1992), pp. 580–583.

- Idea:
  1. Evolve individual trajectories $|\phi(t)\rangle$ stochastically
  2. Calculate observables by averaging over trajectories $\overline{\langle\phi(t)|A|\phi(t)\rangle}$

can evolve
to arbitrary l




**Qtraj!**

Omar, H. B., Escobedo, M. Á., Islam, A., Strickland, M., Thapa, S., Vander Griend, P., & Weber, J. H. (2022). *Computer Physics Communications*, *273*, 108266.

**Advantages:**

- Evolve vector of size $N_H$ instead $N_H^2$ density matrix
- Simulation of individual trajectories is **embarrassingly parallel**

# Connecting with pheno

- Initial state: Localized gaussian peak
- temperature evolution from hydrodynamics simulation

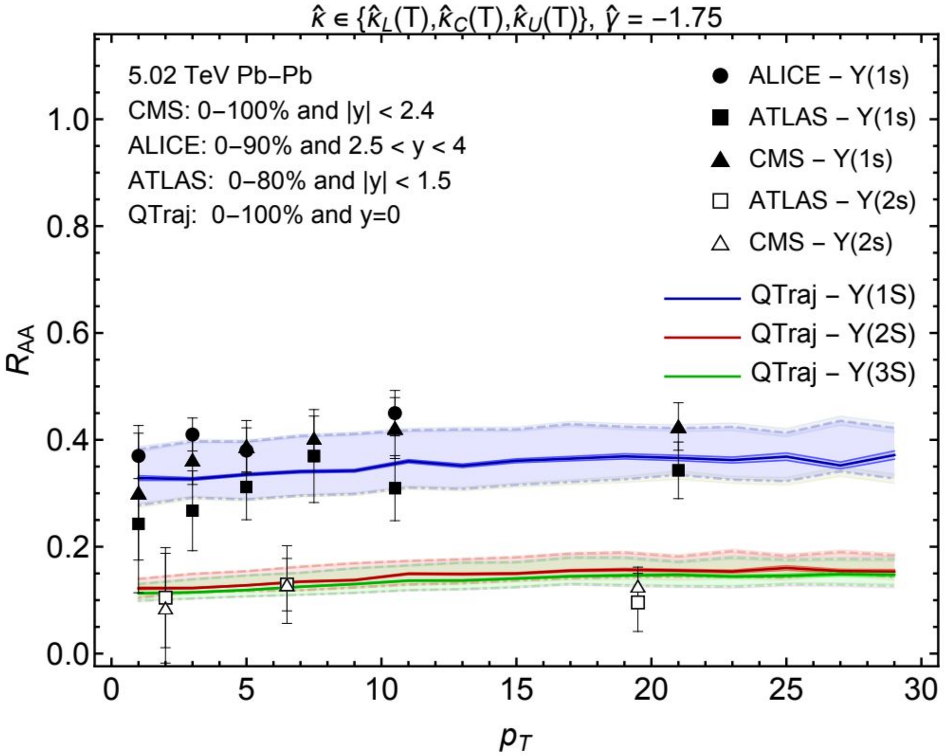
M. Alqahtani and M. Strickland, The European Physical Journal C 81 (2021)

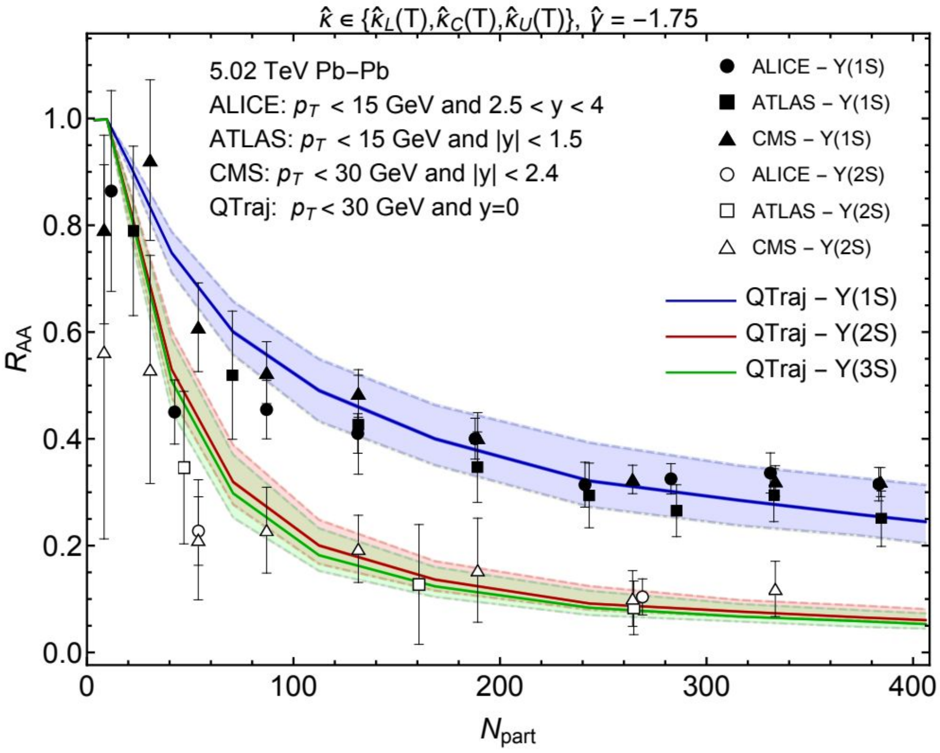
**Survival Probability** $= \dfrac{\langle \psi(t) | 1S \rangle}{\langle \psi(0) | 1S \rangle}$

- Including Feed-down from PDG data

# LO results

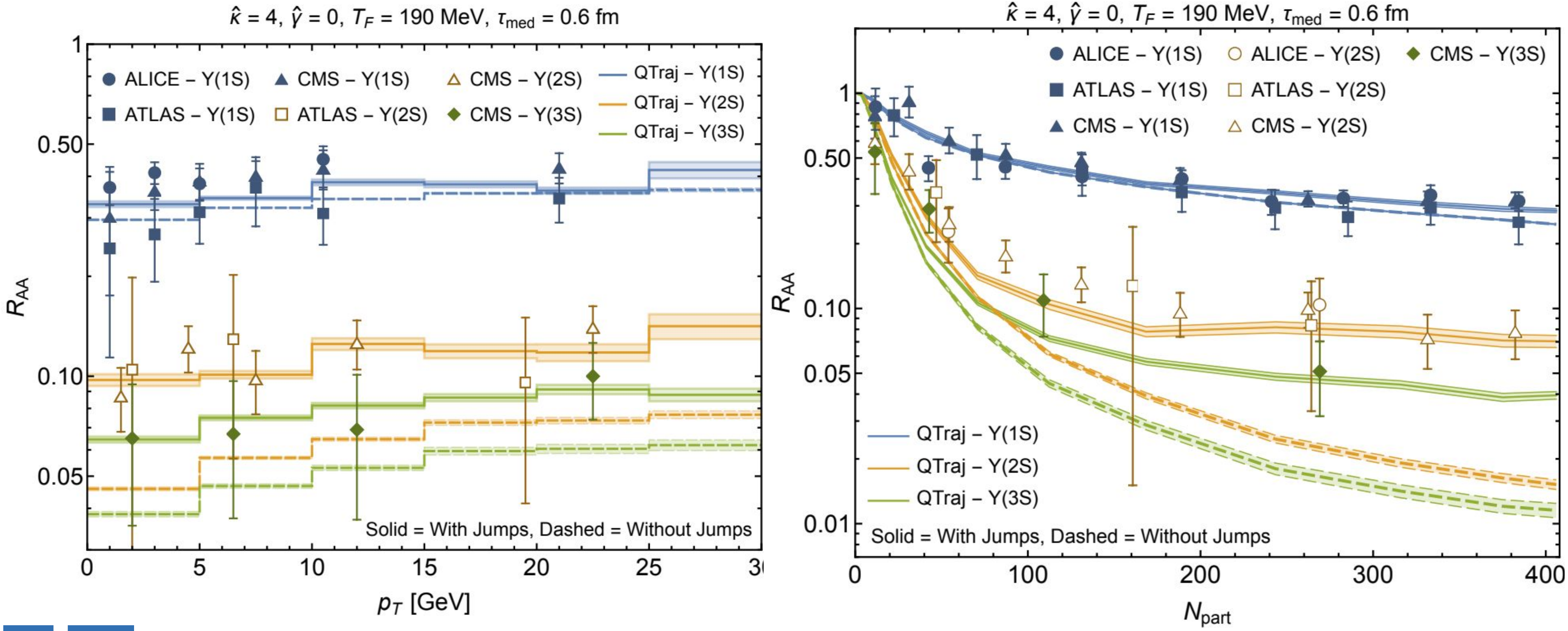
*N. Brambilla, M. A. Escobedo, M. Strickland, A. Vairo, P. Vander Griend and J. H. Weber, Phys. Rev. D 104 (2021) 094049*



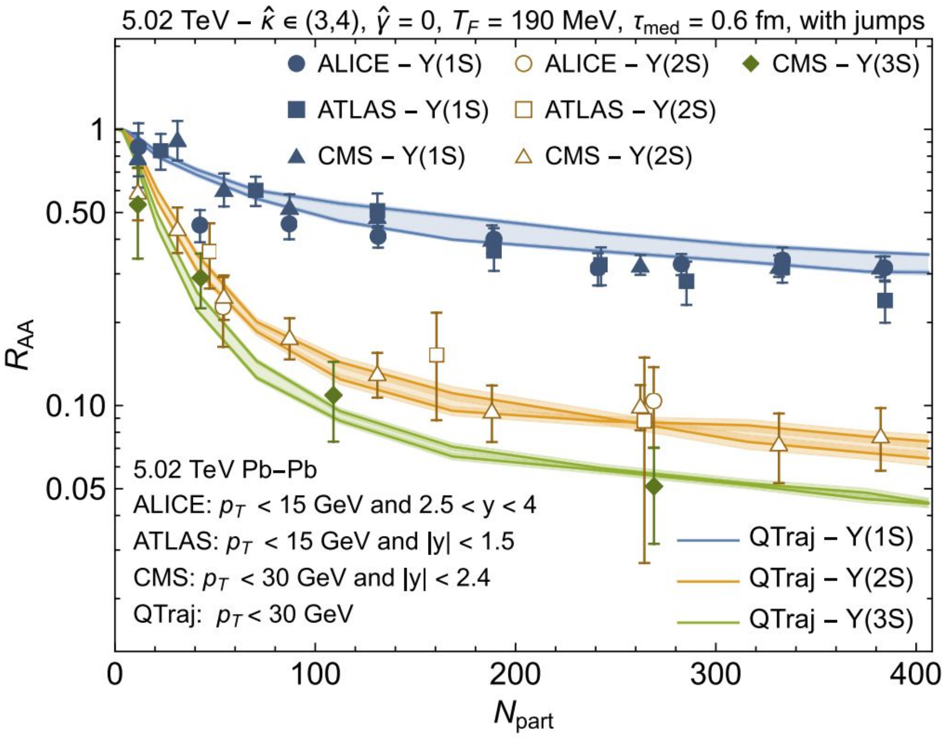
# NLO results

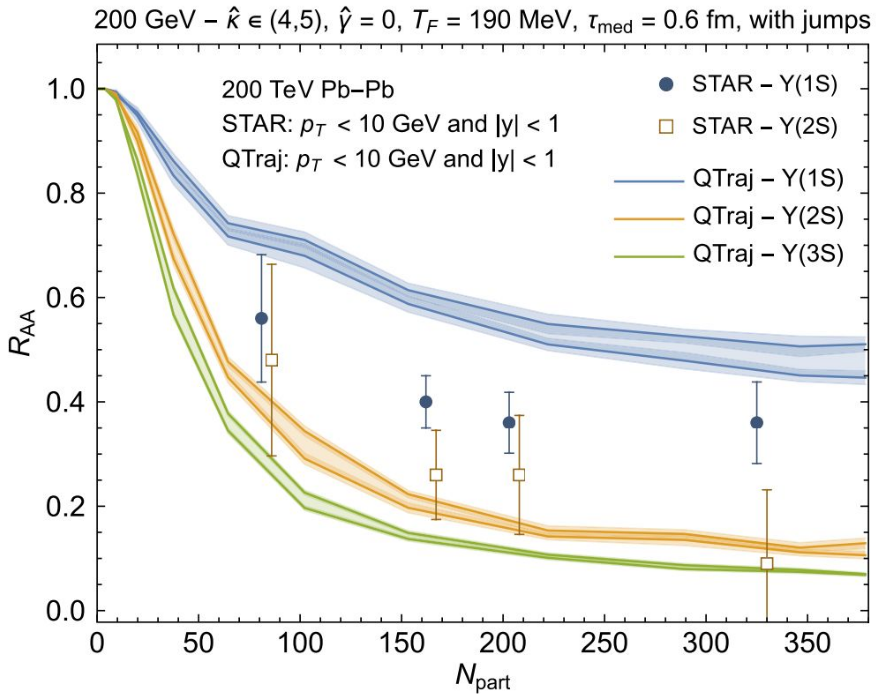
Brambilla, N., Escobedo, M. Á., Islam, A., Strickland, M., Tiwari, A., Vairo, A., & Vander Griend, P. (2023). *Physical Review D*, *108*(1), L011502.



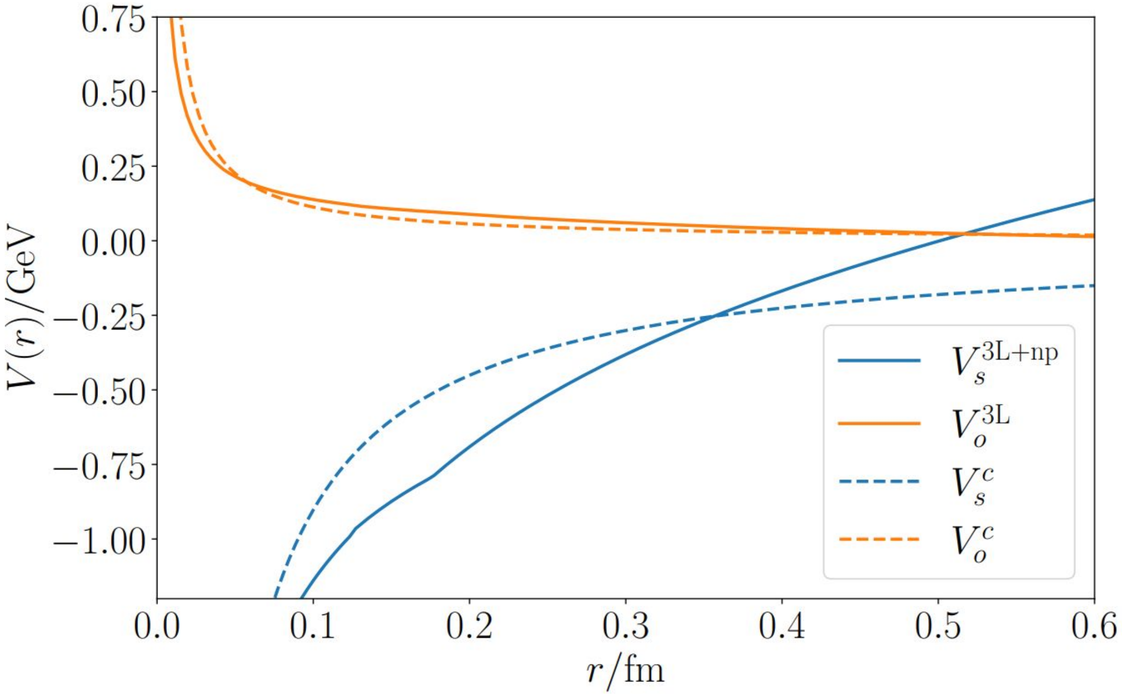
# NLO results

Michael Strickland, Sabin Thapa, PHYSICAL REVIEW D 108, 014031 (2023)



# New Potential



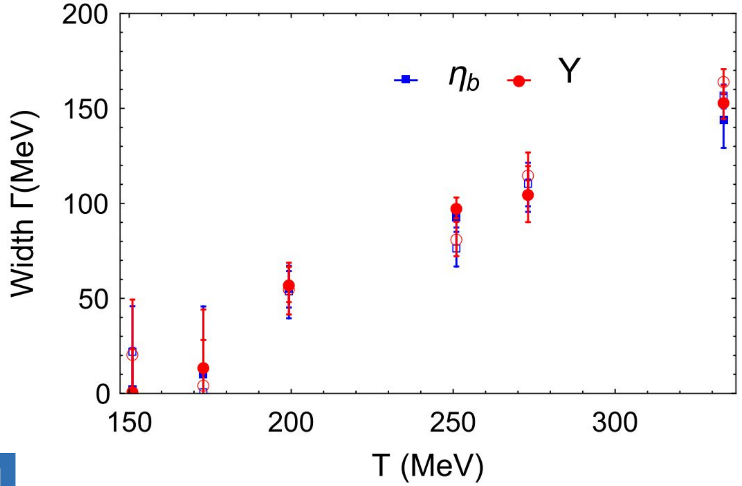

Spectrum:

| | PDG | $V_s^c$ | $V_s^{3\mathrm{L}}$ |
|---|---|---|---|
| $M(1S)/\,\mathrm{GeV}$ | 9.445 | 9.445 | 9.445 |
| $M(2S)/\,\mathrm{GeV}$ | 10.017 | 9.635 | 10.042 |
| $M(3S)/\,\mathrm{GeV}$ | 10.355 | 9.670 | 10.395 |
| $M(1P)/\,\mathrm{GeV}$ | 9.888 | 9.635 | 9.887 |
| $M(2P)/\,\mathrm{GeV}$ | 10.251 | 9.670 | 10.279 |

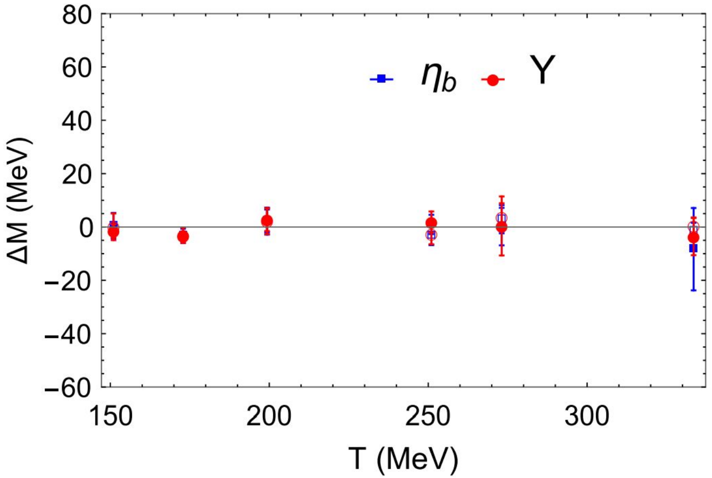
# Determination of transport coefficients

- Indirectly determine $\hat{\kappa}$ and $\hat{\gamma}$ from lattice measurements of the **in medium width** $\Gamma$ and **mass shift** $\delta m$

$$\Gamma \propto \hat{\kappa}$$




$$\delta m \propto \hat{\gamma} \Rightarrow \hat{\gamma} = 0$$

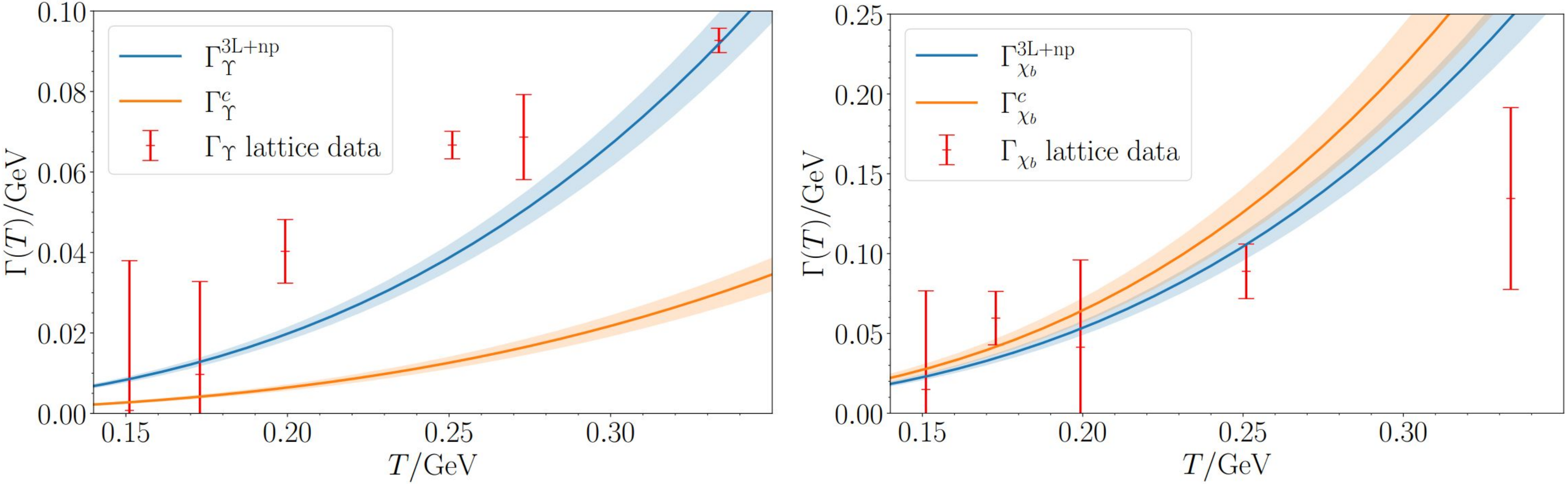



R. Larsen, S. Meinel, S. Mukherjee, and P. Petreczky, Physical Review D 100 (2019)

**see also** A. Andronic,, P.B. Gossiaux , P. Petreczky, R. Rapp, M. Strickland, et al. Eur.Phys.J.A 60 (2024) 4, 88

# Determination of transport coefficients

- Obtain $\hat{\kappa}$ from fits to $1S$ and $1P$ data and average

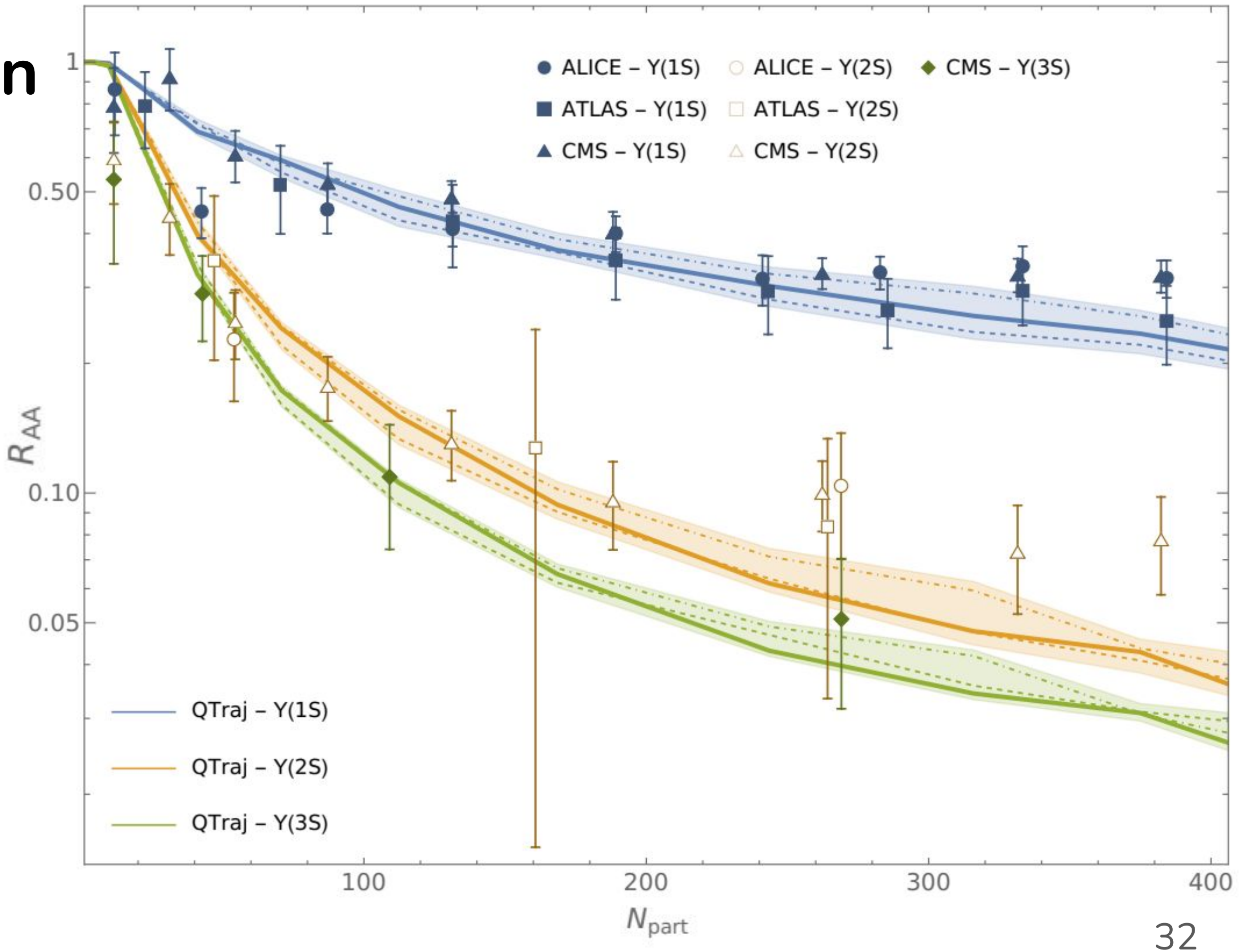



**Coulomb:** $\hat{\kappa} = 0.33 \pm 0.04$

**New potential:** $\hat{\kappa} = 1.88 \pm 0.16$

# Nuclear modification factor results

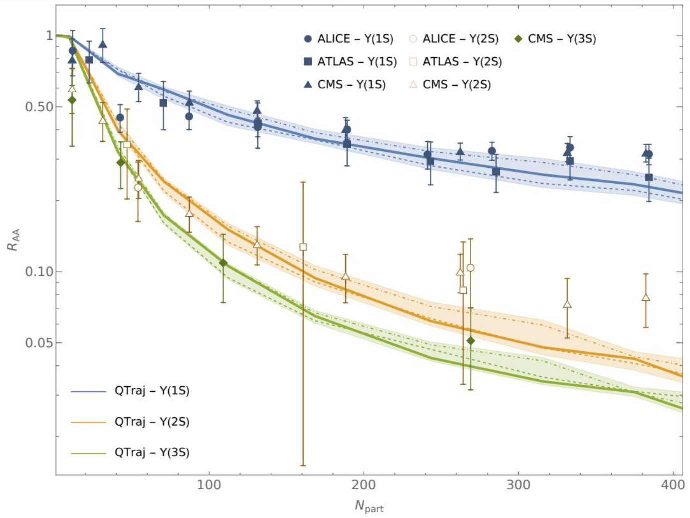
- New potential can describe the experimental data
- Coulomb potential with $\hat{\kappa} = 0.33 \pm 0.04$ can not describe the data

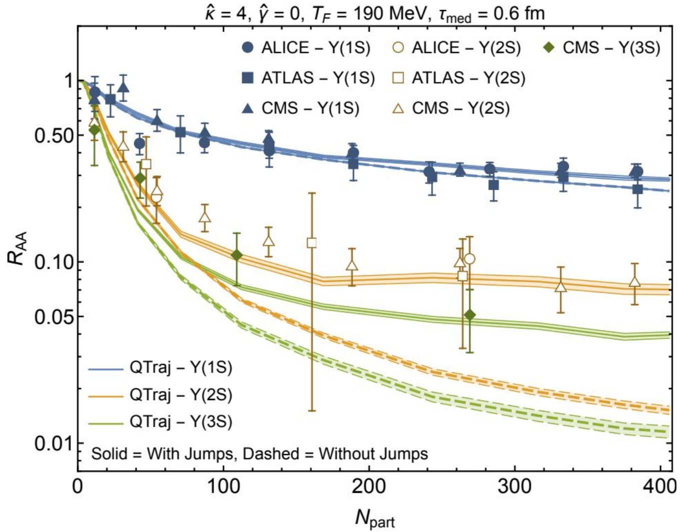


# Summary & Future challenges

- Progressing towards a more and more mature simulation framework

- Challenge assumptions:
  1. Markovian approximation
  2. Assumption of isotropy
  3. Estimate uncertainties form E/T expansion
- Charmonium ?



# Backup slides

# Quantum Trajectories

$$U(\Theta) = 1 - iH_{\mathrm{eff}}\delta t$$

$$\psi_0$$

1. Evolve state $|\psi(t)\rangle$ with $U(\Theta)$

$$|\psi(t+\delta t)\rangle = U(\Theta)|\psi(t)\rangle$$

2. Compute norm

$$\langle\psi(t+\delta t)|\psi(t+\delta t)\rangle = 1 - \delta p(\Theta) < 1$$

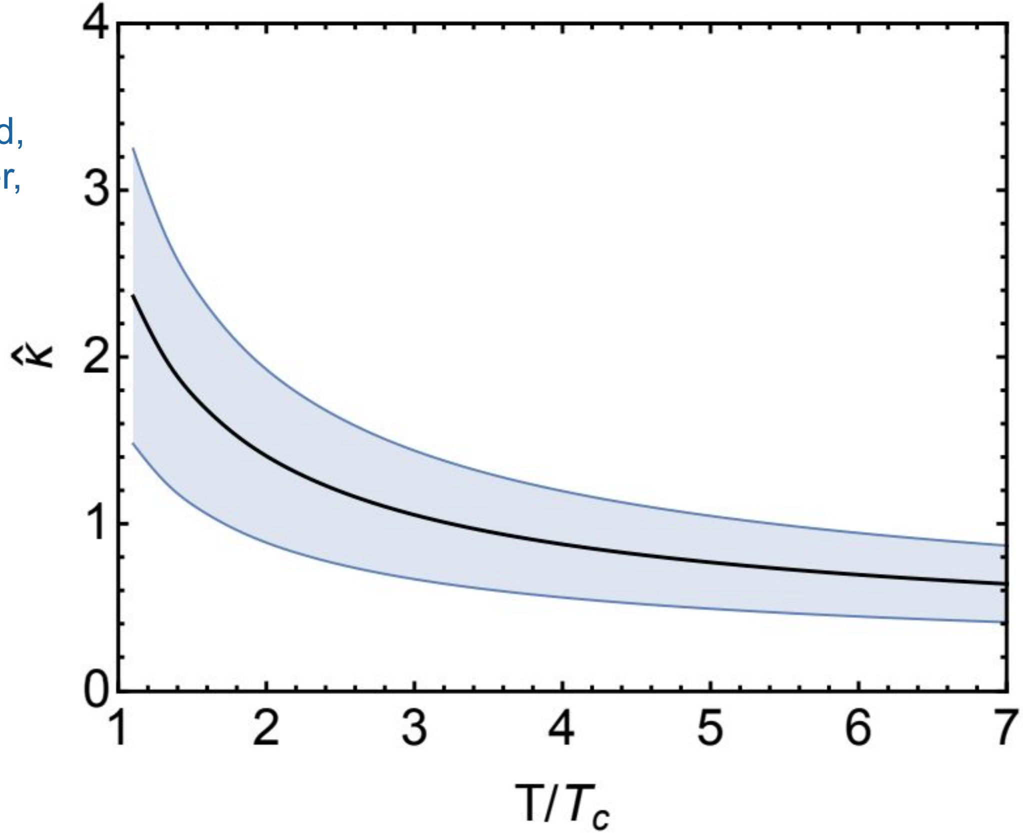
3. Apply jump operator $C(\Theta)$ with probability $\delta p(\Theta)$

$$|\psi(t+\delta t)\rangle = C(\Theta)|\psi(t)\rangle$$

4. Normalize $|\psi(t+\delta t)\rangle$

# Non perturbative correction

N. Brambilla, M. A. Escobedo, M. Strickland, A. Vairo, P. Vander Griend, and J. H. Weber, JHEP 05, 136 (2021), 2012.01240



# Heavy quark diffusion coefficient

$$V_s^{\text{non-pert}}(r) = -i\frac{g^2 T_F}{3N_c} r^2 \int_0^\infty dt \langle E^a(t)\Omega(t,0)^{ab}E^b(0)\rangle$$

$$\gamma = \frac{g^2}{3N_c}\mathrm{Im}\int_0^\infty dt\,\langle E^a(t)\Omega(t,0)^{ab}E^b(0)\rangle$$

# In medium width

- Width given by collapse operators

$$\Gamma = \sum_n C_n^\dagger C_n$$

- At LO in $E/T$

$$\Gamma = \hat{\kappa} T^3 r^2$$

# Determination of transport coefficients

- Indirectly determine $\hat{\kappa}$ and $\hat{\gamma}$ from lattice measurements of the **in medium width** $\Gamma$ and **mass shift** $\delta m$

$$\kappa = \hat{\kappa} T^3$$

no vacuum part

$$\gamma = \gamma(T=0) + \hat{\gamma} T^3$$

- Assume simple model for the vacuum part $\gamma(T=0)$

$$\langle E^a(t)\Omega(t,0)^{ab}E^b(0)\rangle = \langle E^2(0)\rangle \mathrm{e}^{-i\Lambda_E t}$$

G. S. Bali and A. Pineda, Physical Review D 69 (2004)

$$\langle g^2 E^2(0)\rangle = -0.2\mathrm{GeV}^4 \qquad \Lambda_E = 1.25\mathrm{GeV}$$

S. Narison, Qcd parameters and sm-high precisions from e+e-→hadrons :Summary(2023),2309.05342

# EFTs for Quarkonium Suppression

- Use NREFTs to exploit hierarchy of scales

$$M \gg 1/a_0 \gg \pi T \gg E$$

- Inverse radius:

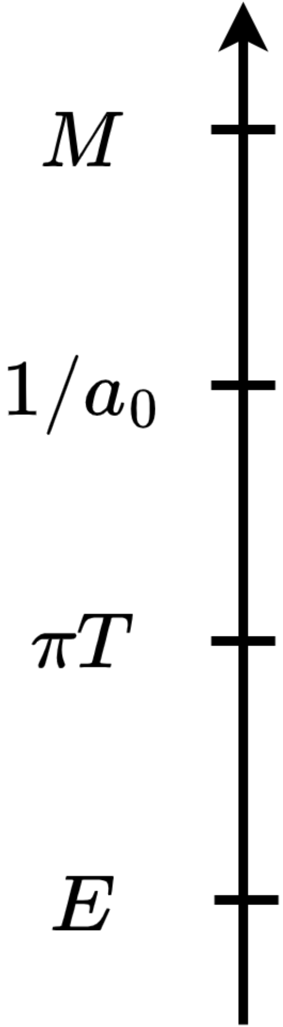
$$1/a_0 \approx 1.2\text{GeV}$$

- Temperature regime:

$$250\text{MeV} < T < 425\text{MeV}$$

- Binding Energy:

$$E \sim 0.4\text{GeV}$$



# New Potential

Kiyo, Y., Pineda, A., & Signer, A. (2010). *Nuclear Physics B*, *841*(1-2), 231-256.

- Motivation: Implement a higher order potential with a more realistic spectrum

$$V_s^{3\mathrm{L}}(r) = V_s^{\mathrm{pert}}(r) + V_s^{\text{non-pert}}(r)$$

$$V_s^{\mathrm{pert}}(\nu, \nu_r, r) = \begin{cases} \sum_{k=0}^{3} V_{s,\mathrm{RS}'}^{(k)} \alpha_s^{k+1}(1/r) & \text{if } r < \nu_r^{-1} \\ \sum_{k=0}^{3} V_{s,\mathrm{RS}'}^{(k)} \alpha_s^{k+1}(\nu) & \text{if } r > \nu_r^{-1} \end{cases}$$

three loop pNRQCD

$$\mathrm{Re}\left(V_s^{\text{non-pert}}(r)\right) = \frac{\gamma}{2} r^2$$

leading non-perturbative correction