Universität Münster

International Workshop "QCD challenges from pp to AA collisions"

Causality conditions as constraints in hydrodynamics

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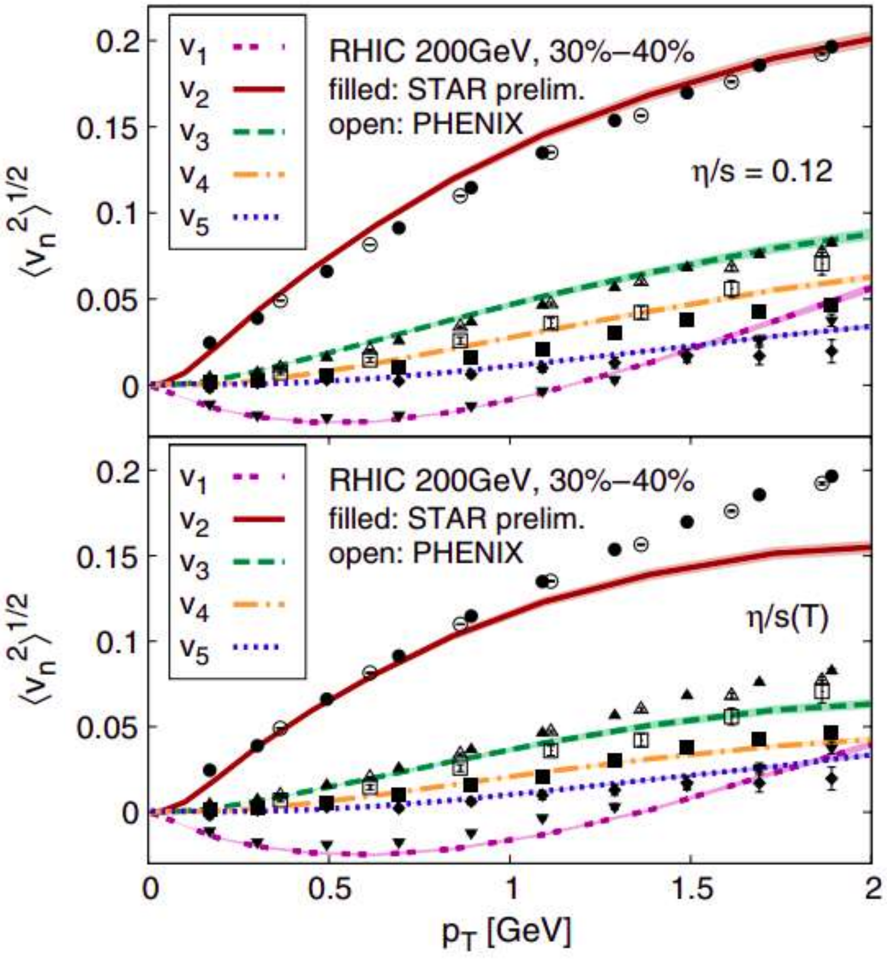
UNIVERSITÄT BIELEFELD

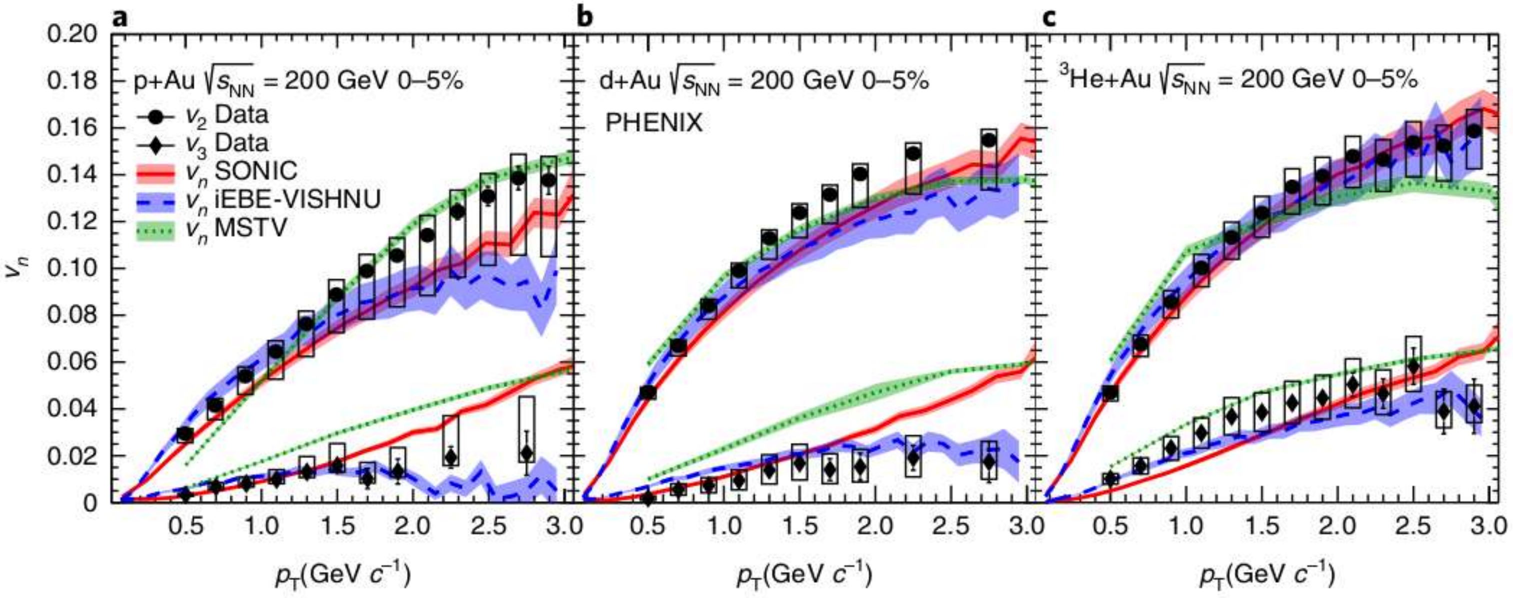
CRC-TR 211
Strong-interaction matter under extreme conditions

Hydrodynamics in heavy Ion Collisions: Defining Its Limits

$$\frac{dN}{d\phi} \sim 1 + 2\sum_{n=1}^{\infty} v_n \cos[n(\phi - \Psi_n)]$$



Gale et al., Phys. Rev. Lett. 110, 012302 (2013)



PHENIX Collaboration, Nature Phys. 15, 214 (2019)

Hydrodynamics effectively describes heavy ion collision data, even in out-of-equilibrium conditions. However, what criteria determine the applicability of hydrodynamics?

Outline

1. Relativistic Dissipative Hydrodynamics Theory

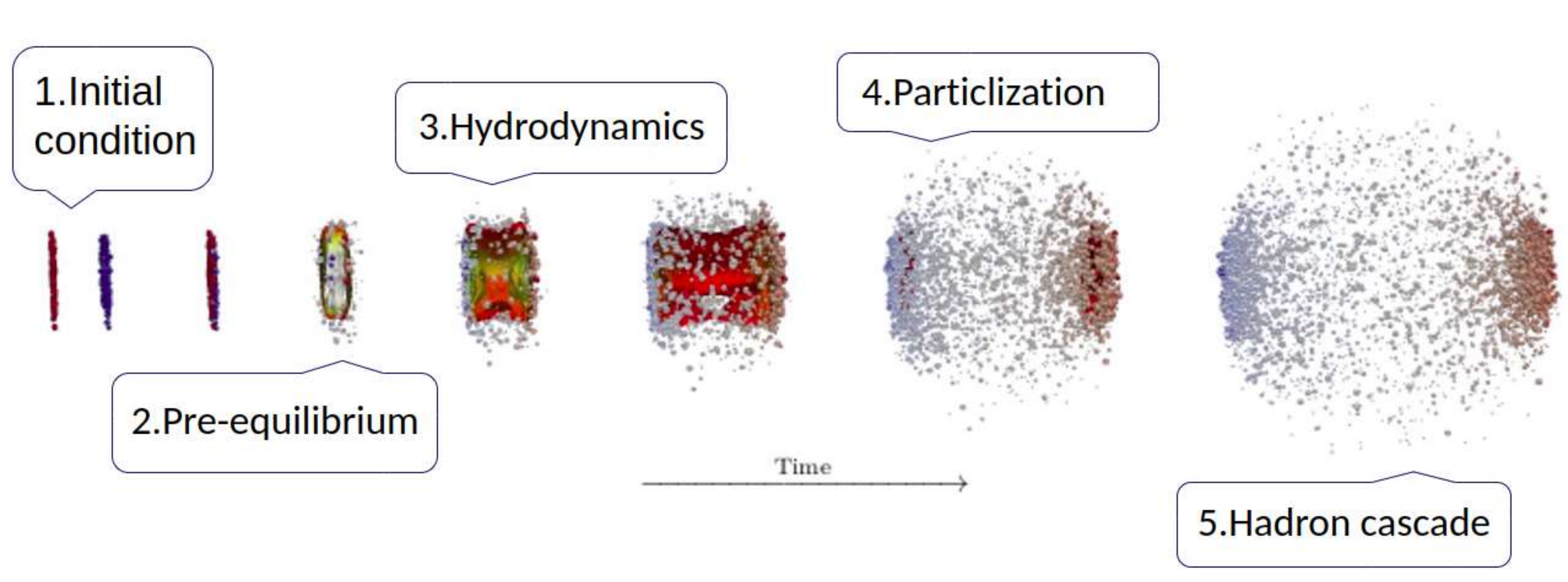
2. Linear Causality Condition

3. Non-linear Causality Conditions

4. Validation Through Simulations

5. Applying Causality Constraints in Bayesian Analysis

1. Relativistic Dissipative Hydrodynamics Theory



Describing the QGP: relativistic, viscous and out of equilibrium

1. Relativistic Dissipative Hydrodynamics Theory

Dissipative fluids: Navier-Stokes theory

Extension to relativistic framework

$$\nabla_\mu T^{\mu\nu} = 0$$

Shear and Bulk →

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu}$$
$$\Pi = -\zeta\nabla^{\perp}_{\lambda}u^{\lambda}$$

Acausal behavior from group velocity in relativistic limits

$$v_g \equiv \left|\frac{d\omega}{dk}\right| = 2\gamma_\eta k \to \infty$$

Maxwell-Cattaneo modifications: incorporating relaxation times

$$\tau_\pi D\pi^{\mu\nu} + \pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \dots$$
$$\tau_\Pi D\Pi + \Pi = -\zeta\nabla^{\perp}_{\mu}u^{\mu} + \dots$$

1. Relativistic Dissipative Hydrodynamics Theory

Heavy ion collisions use Israel-Stewart-type second-order viscous hydrodynamics: DNMR

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\alpha^{\langle\mu}\omega^{\nu\rangle\alpha} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \varphi_7 \pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma^{\nu\rangle\alpha} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + \varphi_6 \Pi\pi^{\mu\nu}$$

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \varphi_1 \Pi^2 + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} + \varphi_3 \pi^{\mu\nu}\pi_{\mu\nu}$$

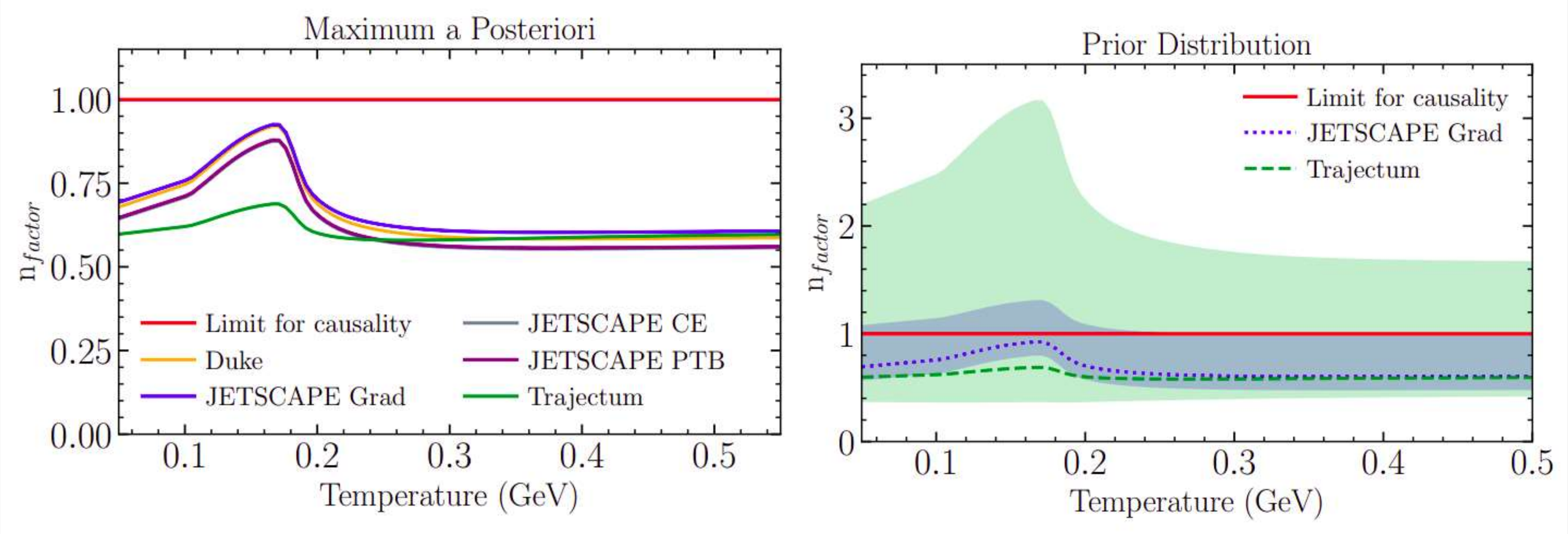
11 second order transport coefficients $\tau_\pi, \delta_{\pi\pi}, \phi_7, \tau_{\pi\pi}, \lambda_{\pi\Pi}, \phi_6, \tau_\Pi, \delta_{\Pi\Pi}, \phi_1, \lambda_{\Pi\pi}, \phi_3$

Linear condition for causality:

$$c_s^2 + \frac{4}{3}\frac{\eta}{\tau_\pi(\epsilon + p)} + \frac{\zeta}{\tau_\Pi(\epsilon + p)} \leq 1$$

2. Linear Causality Condition

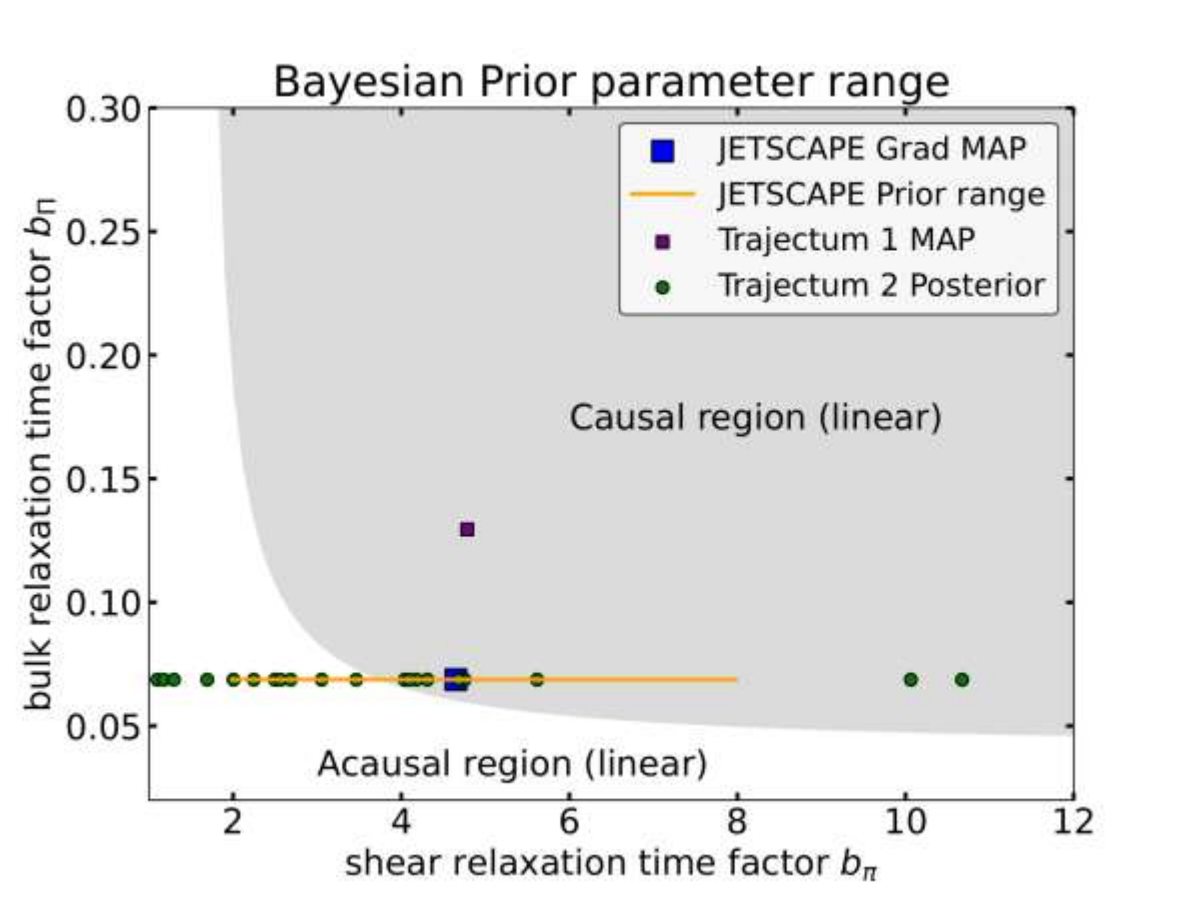
$$c_s^2 + \frac{4}{3}\frac{\eta}{\tau_\pi(\epsilon + p)} + \frac{\zeta}{\tau_\Pi(\epsilon + p)} \leq 1$$



Krupczak et al., Phys. Rev. C, 2024

2. Linear Causality Condition

$$c_s^2 + \frac{4}{3}\frac{\eta}{\tau_\pi(\epsilon + p)} + \frac{\zeta}{\tau_\Pi(\epsilon + p)} \leq 1$$



3. Non-linear Causality Conditions

Necessary conditions

$$
\begin{aligned}
&\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{1}{2}\tau_{\pi\pi}\left|\Lambda_1\right|\geq 0,\\
&\varepsilon+P+\Pi-\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{4\tau_\pi}\Lambda_3\geq 0,\\
&\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)+\frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_a+\Lambda_d\right)\geq 0,\quad a\neq d,\\
&\varepsilon+P+\Pi+\Lambda_a-\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{4\tau_\pi}\left(\Lambda_d+\Lambda_a\right)\geq 0,\quad a\neq d\\
&\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)+\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d+\frac{1}{6\tau_\pi}\left[2\eta+\lambda_{\pi\Pi}\Pi+\left(6\delta_{\pi\pi}-\tau_{\pi\pi}\right)\Lambda_d\right]\\
&\quad+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi}+\left(\varepsilon+P+\Pi+\Lambda_d\right)c_s^2\geq 0\\
&\varepsilon+P+\Pi+\Lambda_d-\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_d-\frac{1}{6\tau_\pi}\left[2\eta+\lambda_{\pi\Pi}\Pi+\left(6\delta_{\pi\pi}-\tau_{\pi\pi}\right)\Lambda_d\right]\\
&\quad-\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_d}{\tau_\Pi}-\left(\varepsilon+P+\Pi+\Lambda_d\right)c_s^2\geq 0
\end{aligned}
$$

Sufficient conditions

$$
\begin{aligned}
&\left(\varepsilon+P+\Pi-\left|\Lambda_1\right|\right)-\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\geq 0\\
&\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\tau_{\pi\pi}\left|\Lambda_1\right|>0,\\
&\tau_{\pi\pi}\leq 6\delta_{\pi\pi},\\
&\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\geq 0,\\
&\frac{1}{3\tau_\pi}\left[4\eta+2\lambda_{\pi\Pi}\Pi+\left(3\delta_{\pi\pi}+\tau_{\pi\pi}\right)\Lambda_3\right]+\frac{\zeta+\delta_{\Pi\Pi}\Pi+\lambda_{\Pi\pi}\Lambda_3}{\tau_\Pi}+\left|\Lambda_1\right|+\Lambda_3 c_s^2\\
&\quad+\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_\pi}\left(\frac{\lambda_{\Pi\pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\left(\Lambda_3+\left|\Lambda_1\right|\right)^2}{\varepsilon+P+\Pi-\left|\Lambda_1\right|-\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{2\tau_{\pi\pi}}\Lambda_3}\leq\left(\varepsilon+P+\Pi\right)\left(1-c_s^2\right),\\
&\frac{1}{6\tau_\pi}\left[2\eta+\lambda_{\pi\Pi}\Pi+\left(\tau_{\pi\pi}-6\delta_{\pi\pi}\right)\left|\Lambda_1\right|\right]+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}\left|\Lambda_1\right|}{\tau_\Pi}+\left(\varepsilon+P+\Pi-\left|\Lambda_1\right|\right)c_s^2\geq 0,\\
&1\geq\frac{\frac{12\delta_{\pi\pi}-\tau_{\pi\pi}}{12\tau_{\pi\pi}}\left(\frac{\lambda_{\Pi\pi}}{\tau_\Pi}+c_s^2-\frac{\tau_{\pi\pi}}{12\tau_\pi}\right)\left(\Lambda_3+\left|\Lambda_1\right|\right)^2}{\left[\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)-\frac{\tau_{\pi\pi}}{2\tau_\pi}\left|\Lambda_1\right|\right]^2},\\
&\frac{1}{3\tau_\pi}\left[4\eta+2\lambda_{\pi\Pi}\Pi-\left(3\delta_{\pi\pi}+\tau_{\pi\pi}\right)\left|\Lambda_1\right|\right]+\frac{\zeta+\delta_{\Pi\Pi}\Pi-\lambda_{\Pi\pi}\left|\Lambda_1\right|}{\tau_\Pi}+\left(\varepsilon+P+\Pi-\left|\Lambda_1\right|\right)c_s^2\\
&\geq\frac{\left(\varepsilon+P+\Pi+\Lambda_2\right)\left(\varepsilon+P+\Pi+\Lambda_3\right)}{3\left(\varepsilon+P+\Pi-\left|\Lambda_1\right|\right)}\times\left\{1+\frac{2\left[\frac{1}{2\tau_\pi}\left(2\eta+\lambda_{\pi\Pi}\Pi\right)+\frac{\tau_{\pi\pi}}{2\tau_\pi}\Lambda_3\right]}{\varepsilon+P+\Pi-\left|\Lambda_1\right|}\right\},
\end{aligned}
$$

Bemfica et al., Phys. Rev. Lett., 2021

4. Validation Through Simulations

CAUSAL

Necessary ✓
Sufficient ✓

INDETERMINATE

Necessary ✓
Sufficient ✗

ACAUSAL

Necessary ✗
Sufficient ✗

4. Validation Through Simulations

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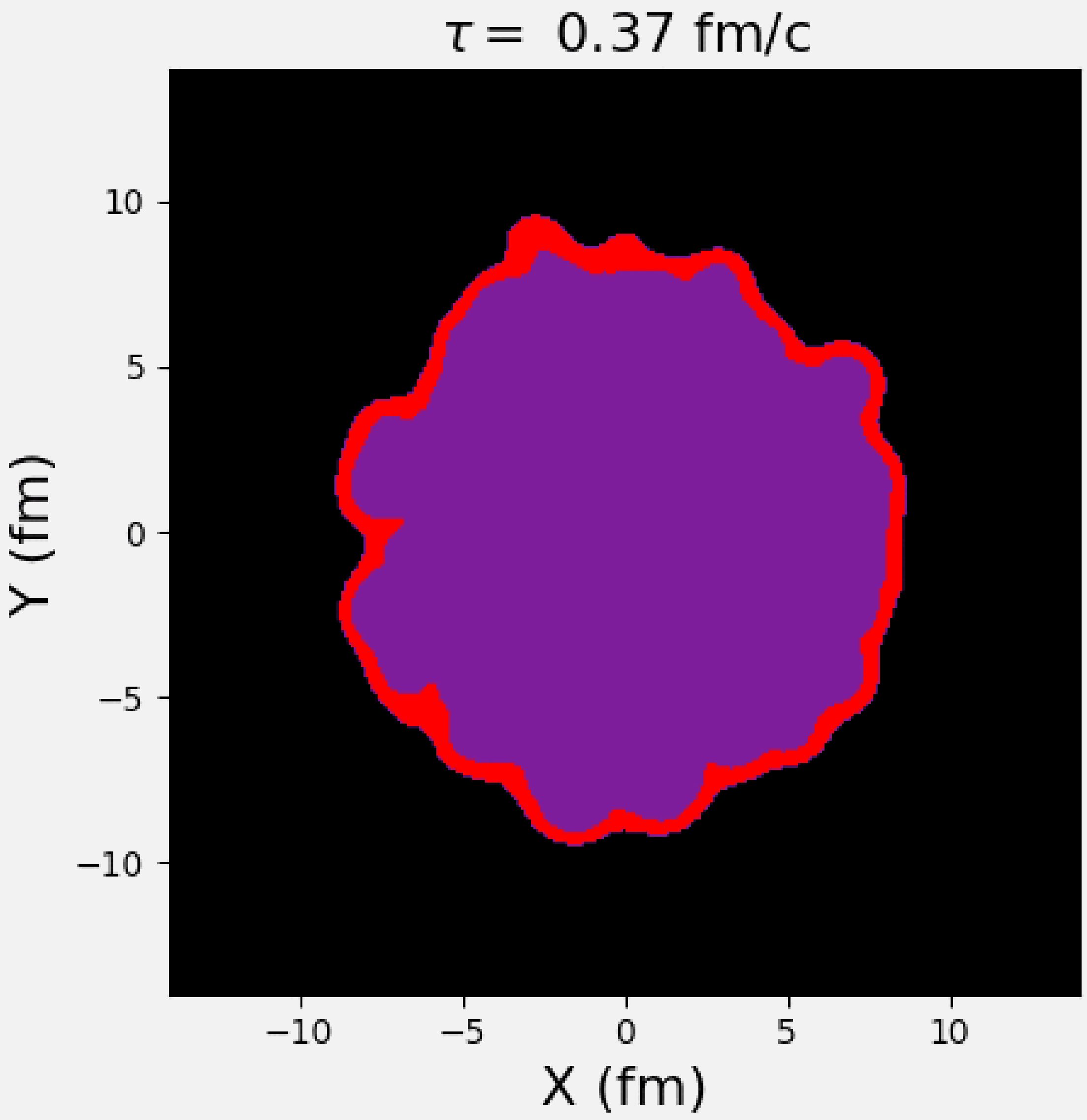

4. Validation Through Simulations

CAUSAL
Necessary ✓
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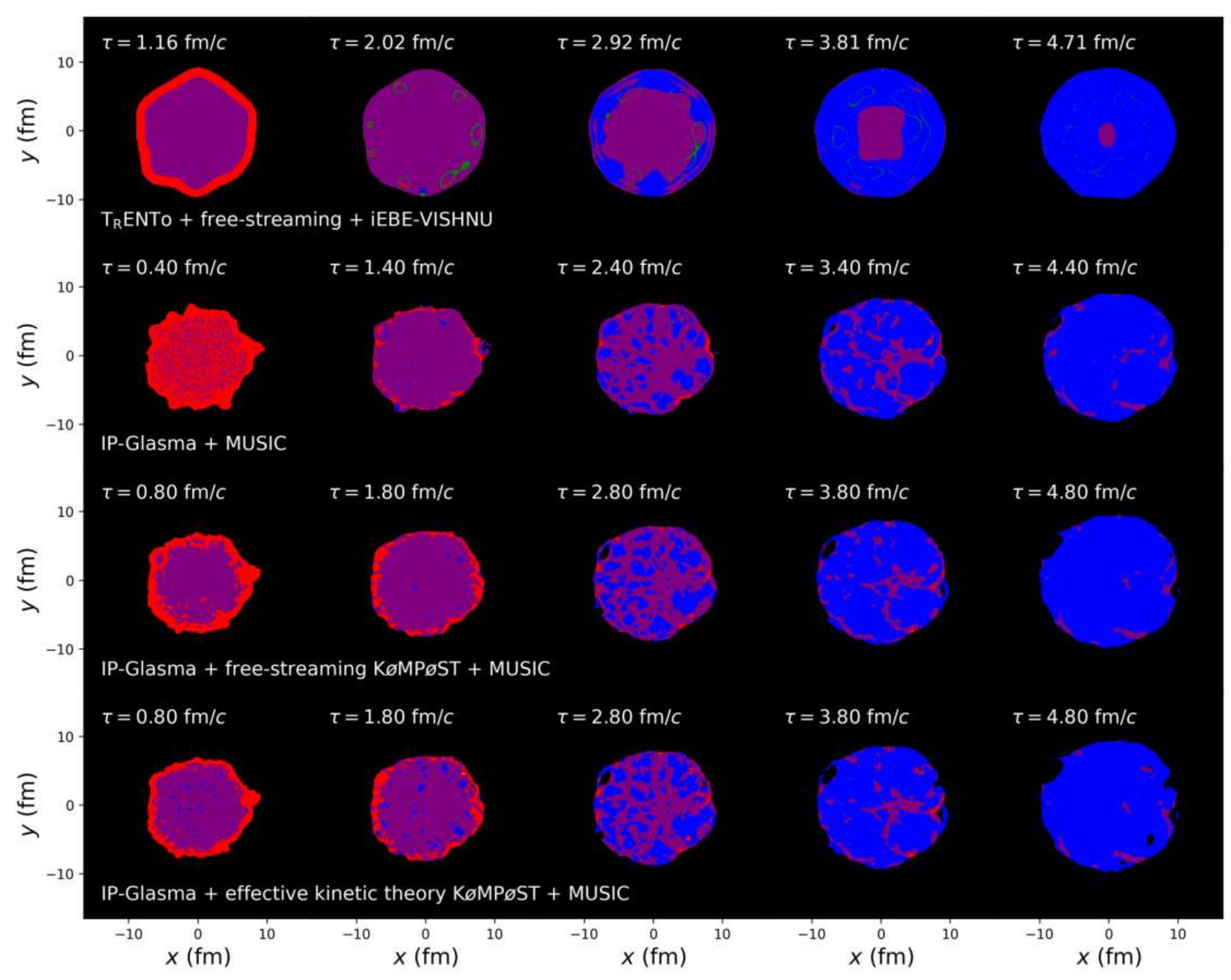
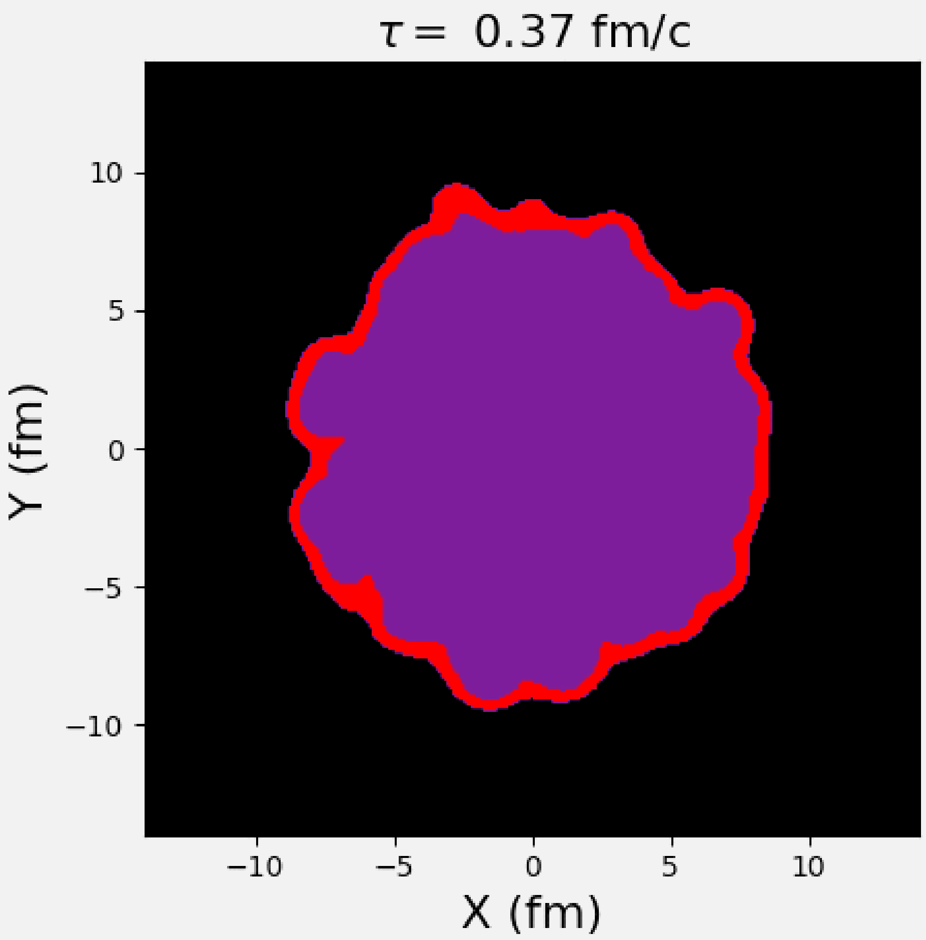


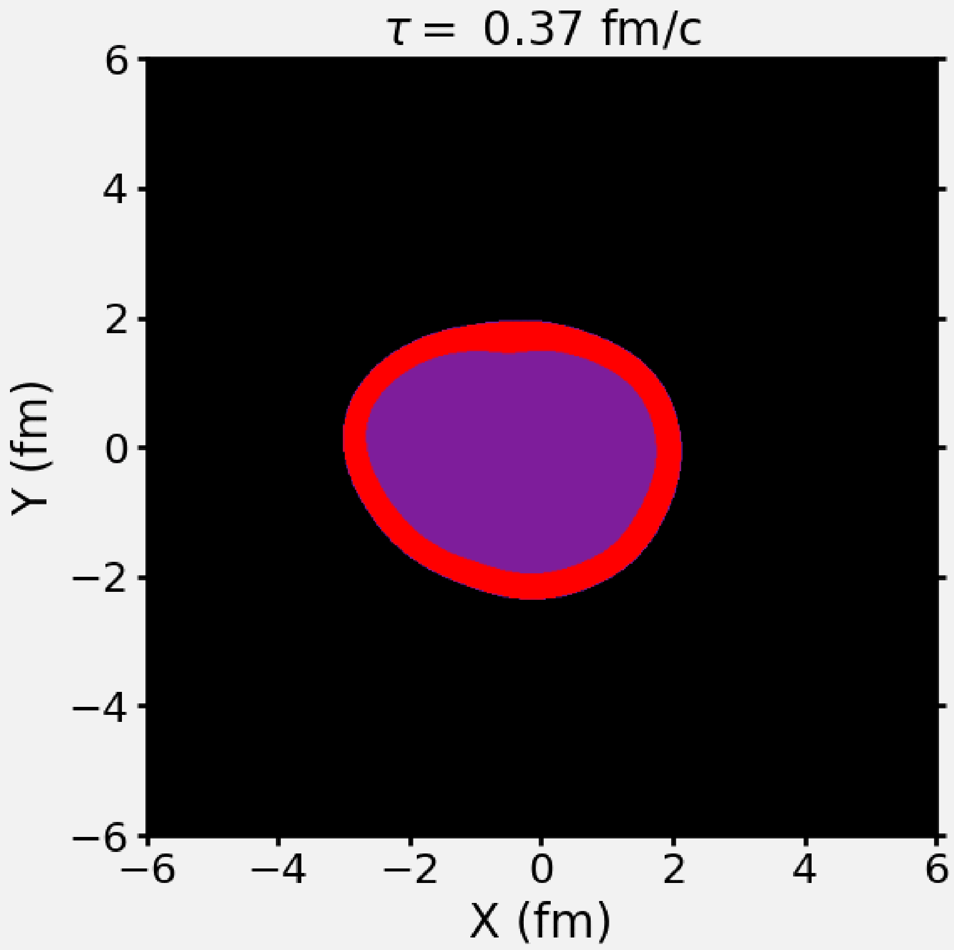
Plumberg et al., Phys. Rev. C, 2022

4. Validation Through Simulations

Comparing large vs. small systems



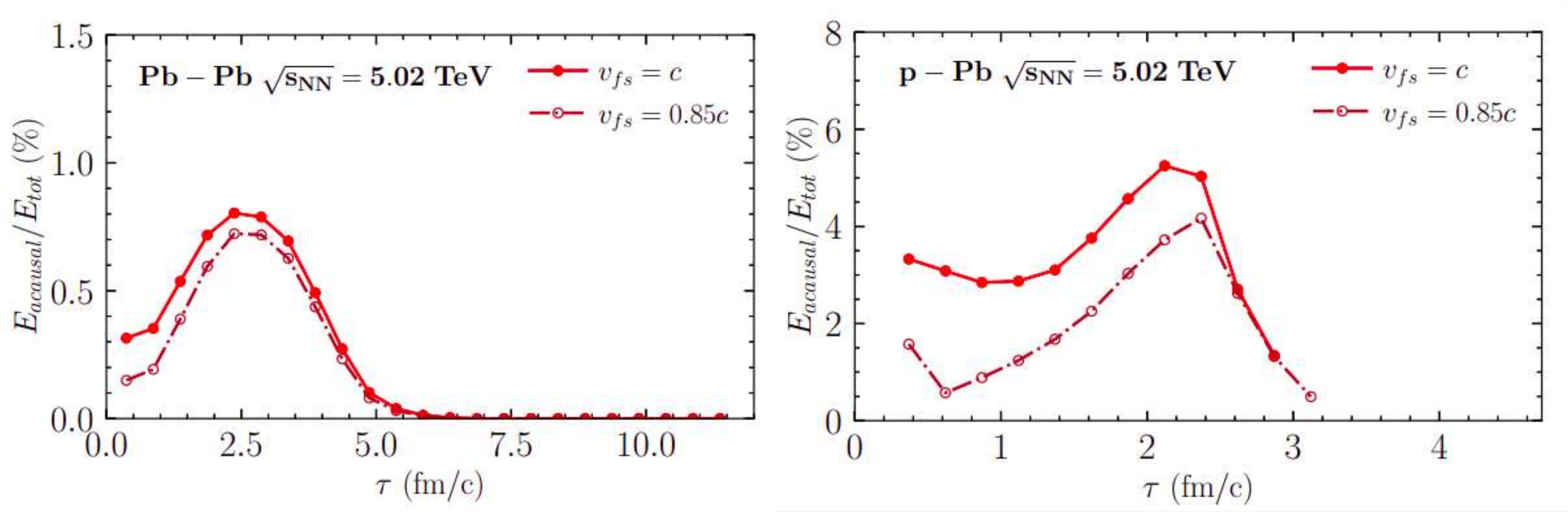
Pb-Pb



p-Pb

4. Validation Through Simulations

Calculating the percentage of acausal energy



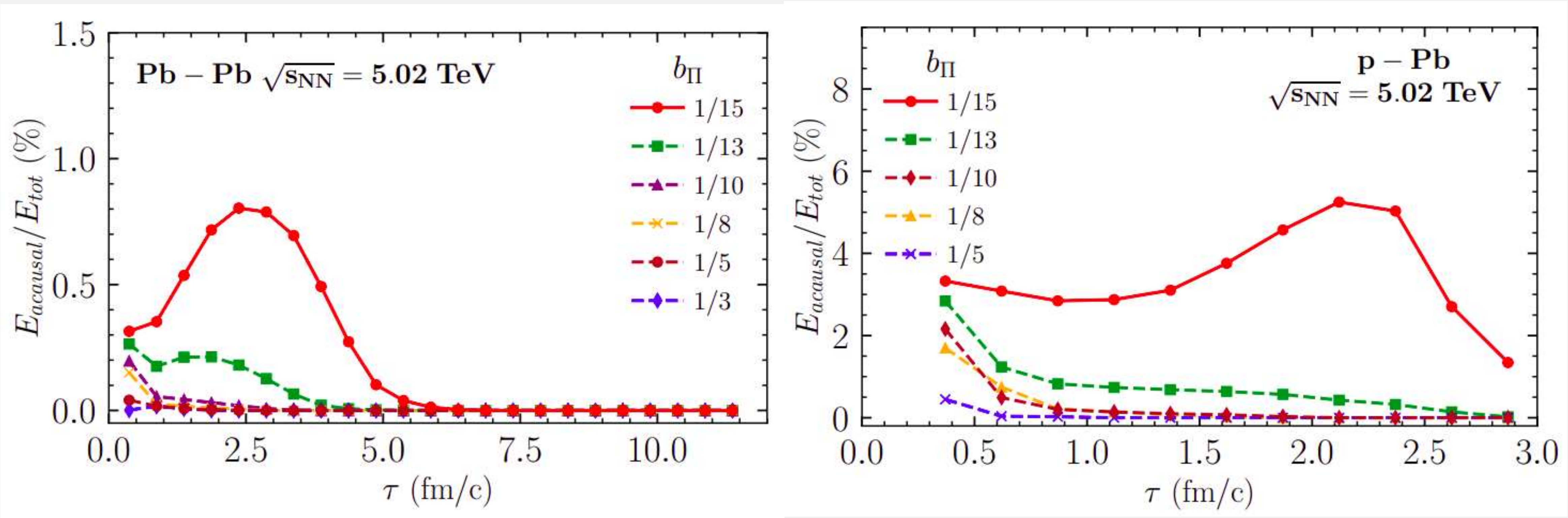
Krupczak et al., Phys. Rev. C, 2024

4. Validation Through Simulations

Reducing acausality by adjusting parameters

$$\tau_\Pi = b_\Pi \frac{\zeta}{(1/3 - c_s^2)^2(\epsilon + P)}$$

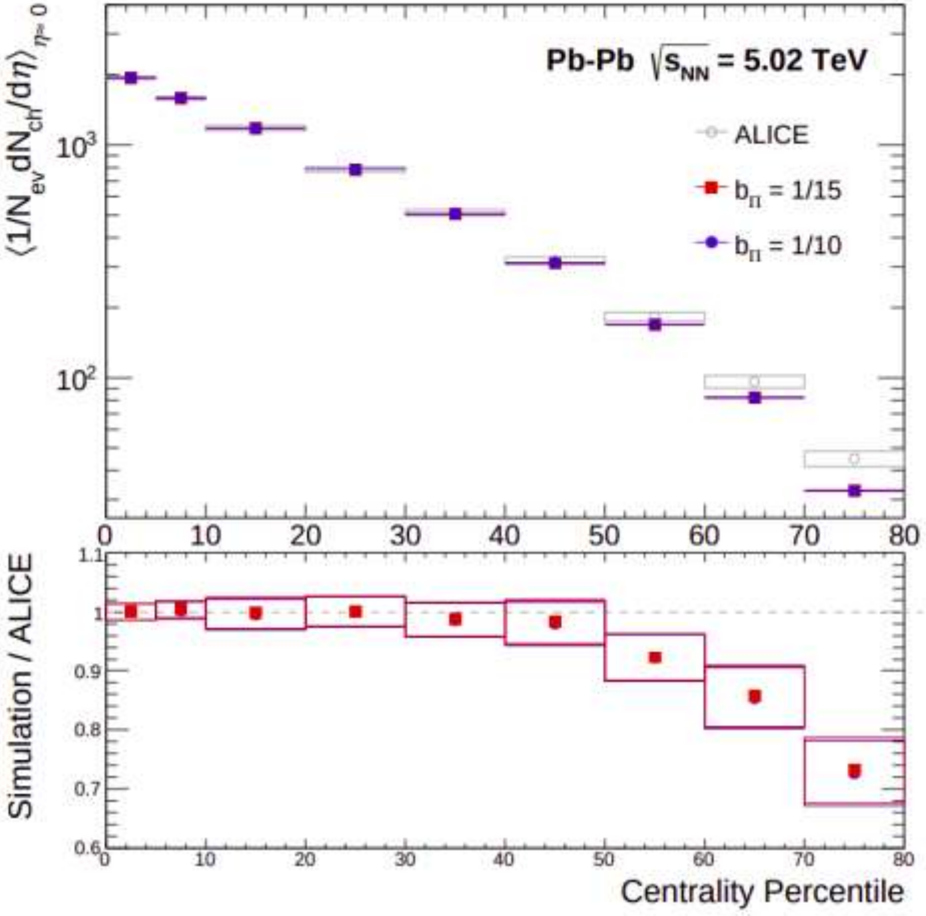
- Dimensionless parameter
- Normalization
- Not determined by hydrodynamic theory
- Does not impact final observables



Krupczak et al., Phys. Rev. C, 2024

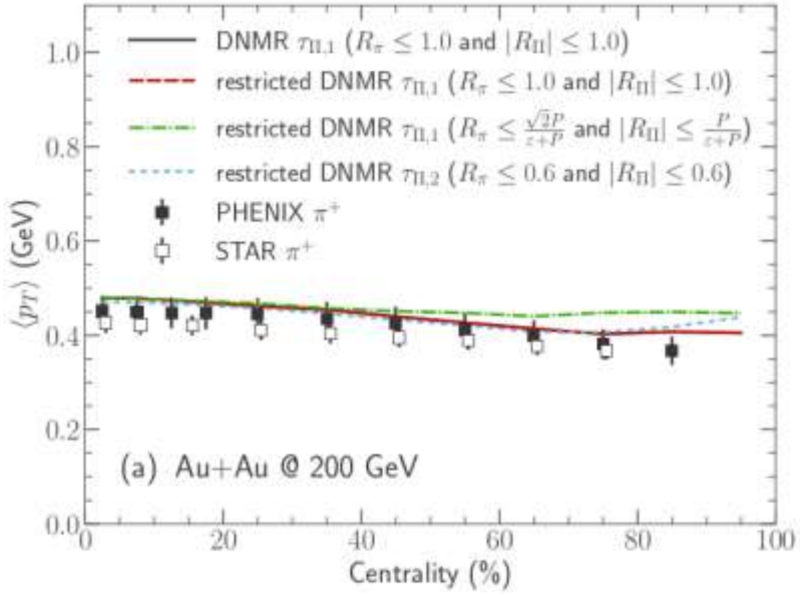
4. Validation Through Simulations

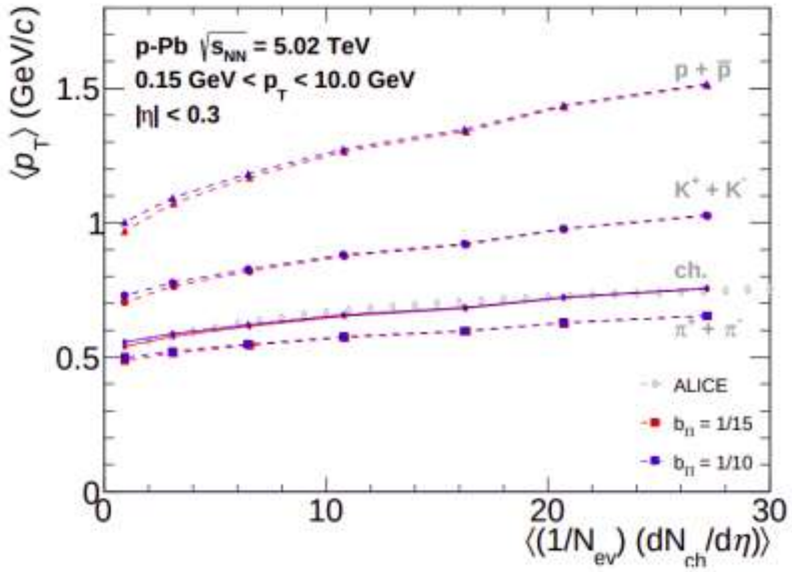
No Impact on final observables



Krupczak et al., Phys. Rev. C, 2024

Chiu & Shen, Phys. Rev. C, 2021





Krupczak et al., Phys. Rev. C, 2024

5. Applying Causality Constraints in Bayesian Analysis

Rewriting the necessary conditions in dimensionless form:

$$n_1 = \frac{2}{C_\eta} + \frac{6}{5}\frac{\Pi}{\varepsilon + P} - \frac{5}{7}\frac{|\Lambda_1|}{\varepsilon + P} \geq 0$$

$$n_2 = 1 - \frac{1}{C_\zeta} + \frac{2}{5}\frac{\Pi}{\varepsilon + P} - \frac{5}{14}\frac{\Lambda_3}{\varepsilon + P} \geq 0$$

$$n_3 = \frac{1}{C_\zeta} + \frac{3}{5}\frac{\Pi}{\varepsilon + P} - \frac{5}{14}\frac{\Lambda_3}{\varepsilon + P} \geq 0$$

$$n_4 = 1 - \frac{1}{C_\eta} + \frac{2}{5}\frac{\Pi}{\varepsilon + P} + \frac{9}{14}\frac{\Lambda_a}{\varepsilon + P} - \frac{5}{14}\frac{\Lambda_d}{\varepsilon + P} \geq 0$$

$$n_5 = c_s^2 + \frac{4}{3C_\eta} + \frac{1}{C_\zeta} + \left(\frac{22}{15} + c_s^2\right)\frac{\Pi}{\varepsilon + P} + \left(\frac{38}{21} + \frac{8(1/3 - c_s^2)}{5} + c_s^2\right)\frac{\Lambda_1}{\varepsilon + P} \geq 0$$

$$n_6 = 1 - c_s^2 - \frac{4}{3C_\eta} - \frac{1}{C_\zeta} + \left(-\frac{7}{15} - c_s^2\right)\frac{\Pi}{\varepsilon + P} + \left(-\frac{17}{21} - \frac{8(1/3 - c_s^2)}{5} - c_s^2\right)\frac{\Lambda_3}{\varepsilon + P} \geq 0$$

Chiu & Shen, Phys. Rev. C, 2021

where

$$C_\eta = \tau_\pi \frac{\epsilon + P}{\eta} = b_\pi$$

$$C_\zeta = \tau_\Pi \frac{\epsilon + P}{\zeta} = b_\Pi \frac{1}{(1/3 - c_s^2)^2}$$

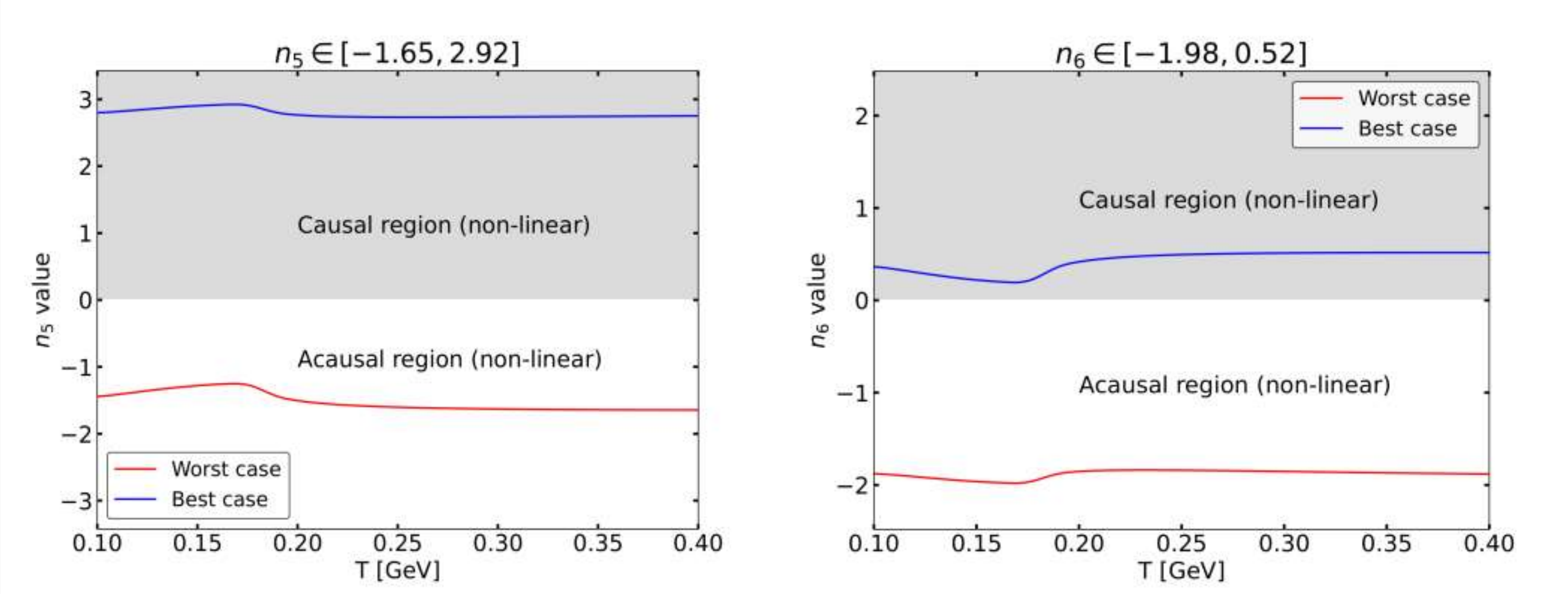
These inequalities depend on relaxation times, eigenvalues of the shear matrix, energy density, pressure, viscosities and sound speed.

We adjust the interval of the prior distribution from JETSCAPE and determine the limits for each equation, always considering the worst-case scenario for eigenvalues.

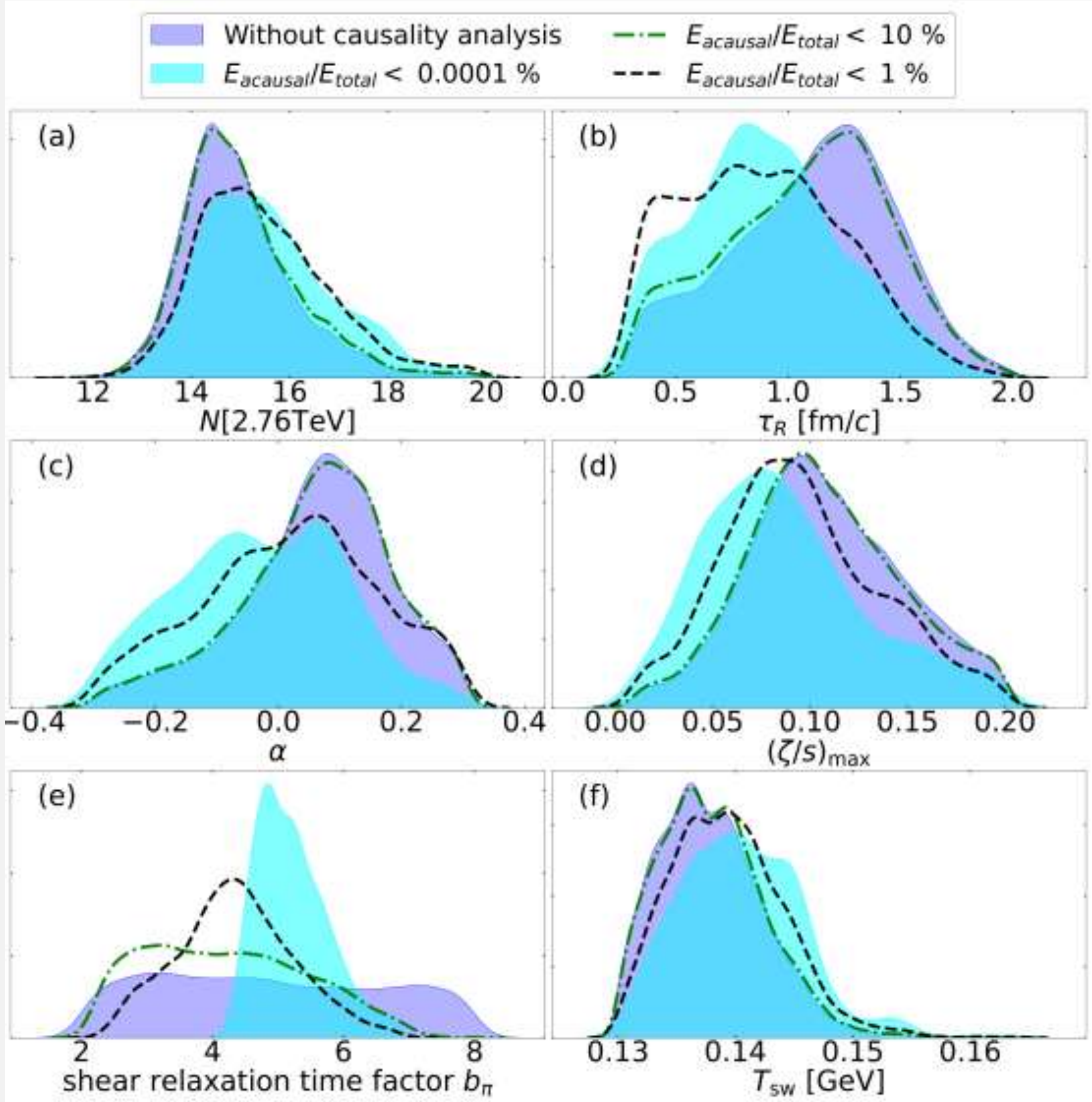
5. Applying Causality Constraints in Bayesian Analysis

In these approximations, we have both temperature-independent and temperature-dependent expressions, which the temperature dependence arising from the sound speed.

$$n_1 \in [-0.46, 2.2]$$
$$n_2 \geq 0 \text{ always}$$
$$n_3 \in [-0.23, 1.1]$$
$$n_4 \in [-0.5, 1.875]$$



5. Applying Causality Constraints in Bayesian Analysis



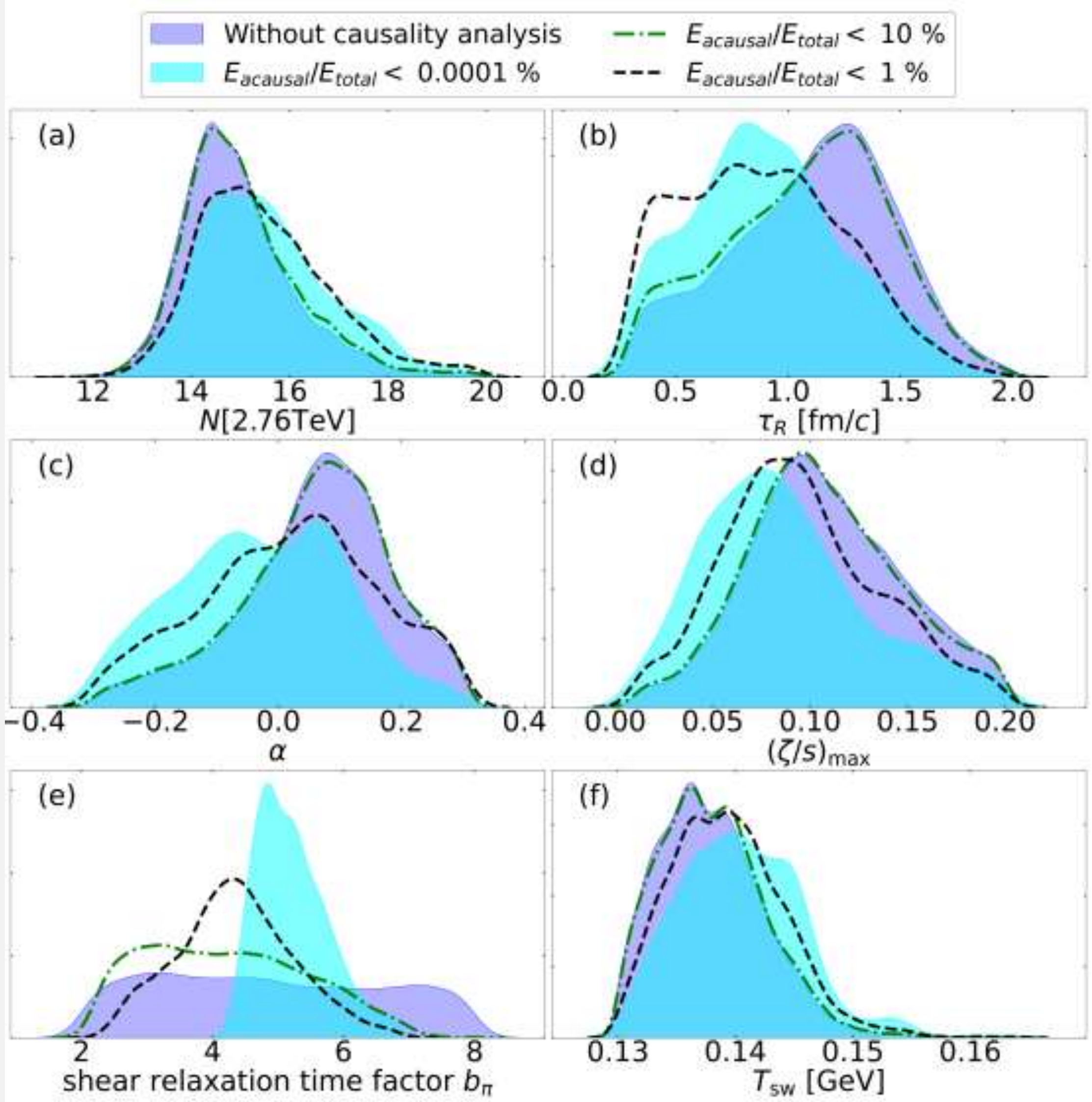
In preparation, 2024

Analysis based on the Prior distribution from JETSCAPE, modified to require an initial acausality of 0.0001% (numerical zero) .

The analysis is conducted using only Trento and free-streaming models, considering all 17 parameters from JETSCAPE due to their correlation.

Posterior for the parameters most affected by the analysis.

5. Applying Causality Constraints in Bayesian Analysis



Posterior for Parameters most affected by the analysis.

- Larger initial normalization;
- Shorter free-streaming duration;

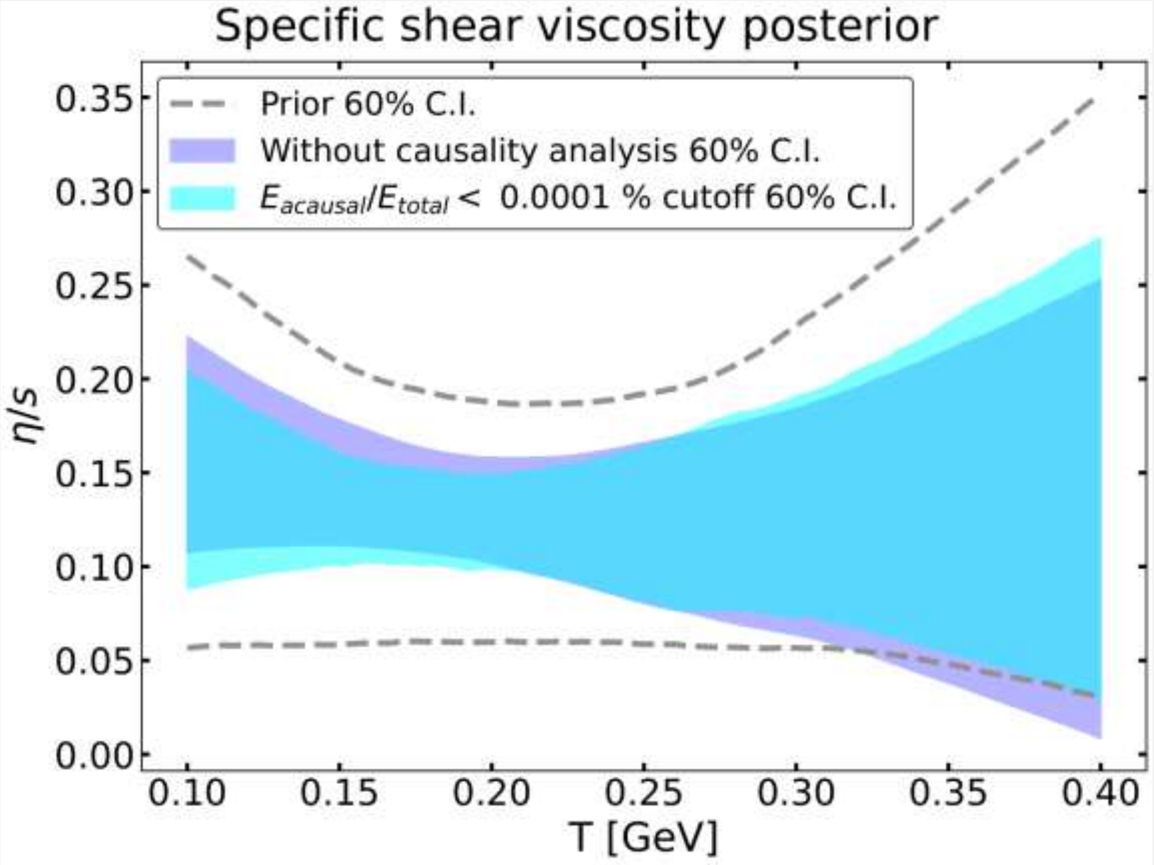
$$\tau_{fs} = \tau_R \left(\frac{\langle \bar{\epsilon} \rangle}{4} \right)^{\alpha}$$

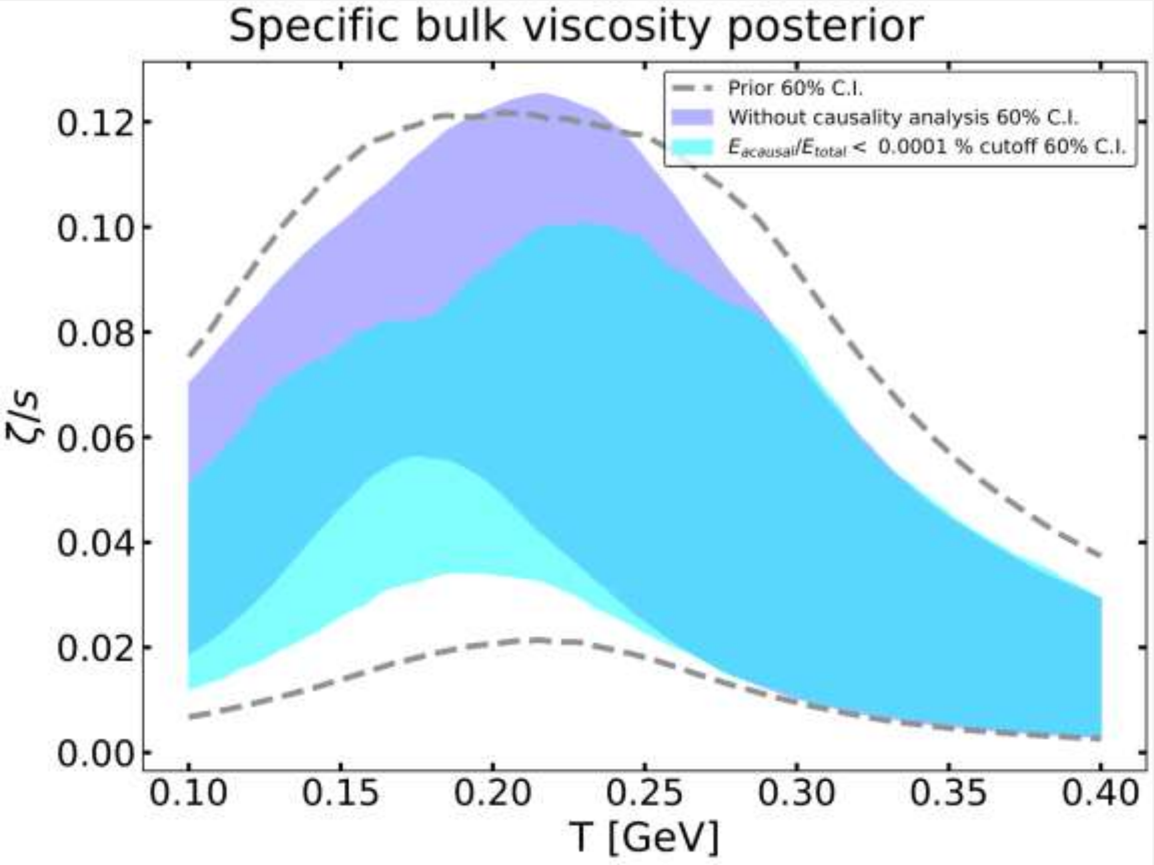
- Reduced bulk viscosity;

$$\frac{\zeta}{s}(T) = \frac{(\zeta/s)_{\max}\Lambda^2}{\Lambda^2 + (T - T_\zeta)^2}$$

- Relaxation time most affected. In the original version, the posterior distribution is nearly flat since this parameter minimally impacts the final observables, consistent with previous articles;
- Higher switching temperature.

5. Applying Causality Constraints in Bayesian Analysis





In preparation, 2024

Conclusions

- Importance of considering the linear causality conditions;
- The impact of causality analysis on the final observables is minimal, making it challenging to assess;
- Relaxation times are the most important parameters, significantly influencing causality conditions;
- Causality conditions can be used to define the applicability limits of hydrodynamics, as the theory must be causal and stable;
- Improvements:

I. Better pre-equilibrium descriptions to initialize hydrodynamics with correct parameters.
II. Studying how regulators inside hydrodynamic codes affect acausality.
III. Performing a comprehensive Bayesian analysis considering causality and varying both relaxation times.
IV. Use these results to define the limit of hydrodynamic applicability for small systems.

Thank you!

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