Stabilizing complex Langevin for real-time gauge theory

Kirill Boguslavski, Paul Hotzy, David I. Müller, [arXiv:2212.08602]

TECHNISCHE UNIVERSITÄT WIEN Vienna | Austria

FWF Der Wissenschaftsfonds.

1 Motivation

- ▹ A **full first-principles description** of the time evolution of the quark-gluon plasma (QGP) in heavy-ion collisions is still missing.
- ▹ Current models and approaches to non-equilibrium QCD successfully explain parts of the QGP evolution but are limited in applicability (class.-stat., kinetic theory, AdS/CFT).
- ▹ Hydrodynamical equations require **transport coefficients** (viscosities).
- ▹ Evolution equations for **hard probes** (e.g., jets, heavy quarks / quarkonia) also need transport coefficients (jet quenching parameter $\hat{q}$, diffusion coefficient $\kappa, \ldots$).
- ▹ Direct computations of such QCD real-time observables are difficult due to the infamous sign problem in $Z = \int DA\, e^{iS[A]}$ stemming from the **real-time path** in Fig. 1.

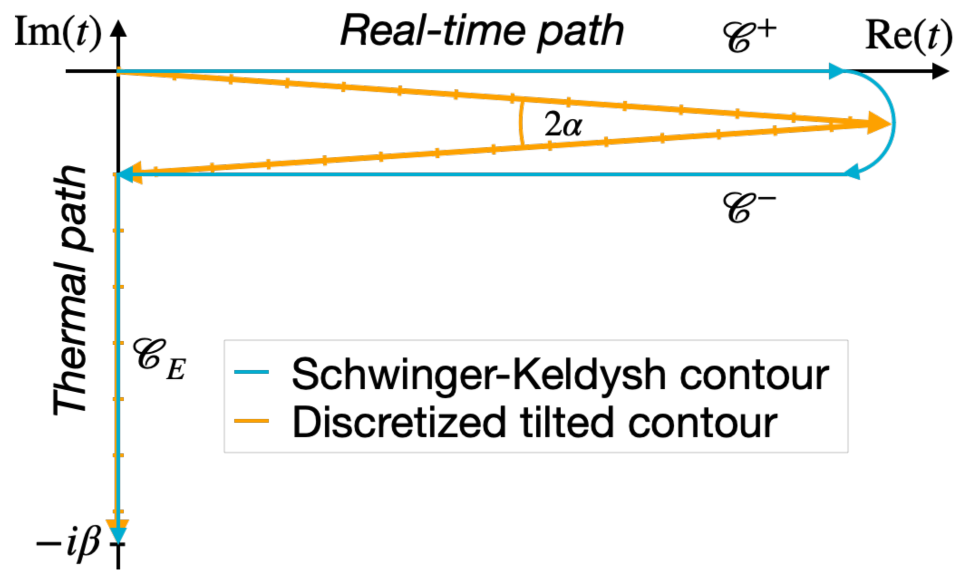

Figure 1: Continuous and discretized Schwinger-Keldysh contour: real-time + thermal (Euclidean) path. Path integral Z regularized by tilting the contour with α.

2 Complex Langevin (CL) for real-time Yang-Mills simulations

- ▹ Approach: **Complex Langevin (CL)** method for Yang-Mills theory (continuum)

$$\frac{\partial A^a_\mu(\theta,x)}{\partial\theta} = i\frac{\delta S_{\rm YM}}{\delta A^a_\mu(\theta,x)} + \eta^a_\mu(\theta,x), \qquad S_{\rm YM} = -\frac{1}{4}\int_{\mathscr{C}} d^4x F^{\mu\nu}_a F^a_{\mu\nu}$$

$$\langle\eta^a_\mu(\theta,x)\rangle = 0, \quad \langle\eta^a_\mu(\theta,x)\eta^b_\nu(\theta',y)\rangle = 2\delta(\theta-\theta')\delta^{(d)}(x-y)\delta^{ab}\delta_{\mu\nu}.$$

- ▹ To compute oscillatory expectation values at sufficiently late Langevin times θ:

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int DA\,\mathcal{O}[A]e^{iS[A]} \approx \lim_{\theta_0\to\infty}\frac{1}{T}\int_{\theta_0}^{\theta_0+T} d\theta\,\mathcal{O}[A(\theta)]$$

- ▹ Complexification of Lie algebra (generators t^a) of the gauge group: $\mathrm{SU}(N)\to\mathrm{SL}(N,\mathbb{C})$
- ▹ **Discretized CL step** (size $N_t N_s^3$, spacings a_μ, $\tilde{A}^a_{x,\mu} = g a_\mu A^a_{x,\mu}$, $U \approx e^{it^a\tilde{A}^a}$, kernel Γ_μ):

$$U_{x,\mu}(\theta+\epsilon) = \exp\left(it^a\left[i\Gamma_\mu\epsilon\frac{\delta S_{\rm W}}{\delta\tilde{A}^a_{x,\mu}} + \sqrt{\Gamma_\mu\epsilon}\,\eta^a_{x,\lambda}(\theta)\right]\right)U_{x,\mu}(\theta) \qquad (1)$$

3 Modern stabilization methods

- ▹ **Adaptive stepsize (AS)** [1] counteracts numerical runaways:

$$\epsilon\mapsto\tilde{\epsilon} = \epsilon\frac{B}{\max\limits_{x,\mu,a}\left|\frac{\delta S_{\rm W}}{\delta\tilde{A}^a_{x,\mu}}\right|}$$

- ▹ **Gauge cooling (GC)** [2] stabilization by reducing 'distance' $F[U]$ to $\mathrm{SU}(N)$:

$$U_{x,\mu}\mapsto U^V_{x,\mu} = V_{x,\mu}U_{x,\mu}V^{-1}_{x+\mu,\mu},$$

$$F[U] = \sum_{x,\mu}\mathrm{Tr}\left[(U_{x,\mu}U^\dagger_{x,\mu}-1)^2\right]\to\min$$

4 Stabilization using new anisotropic kernel Γ

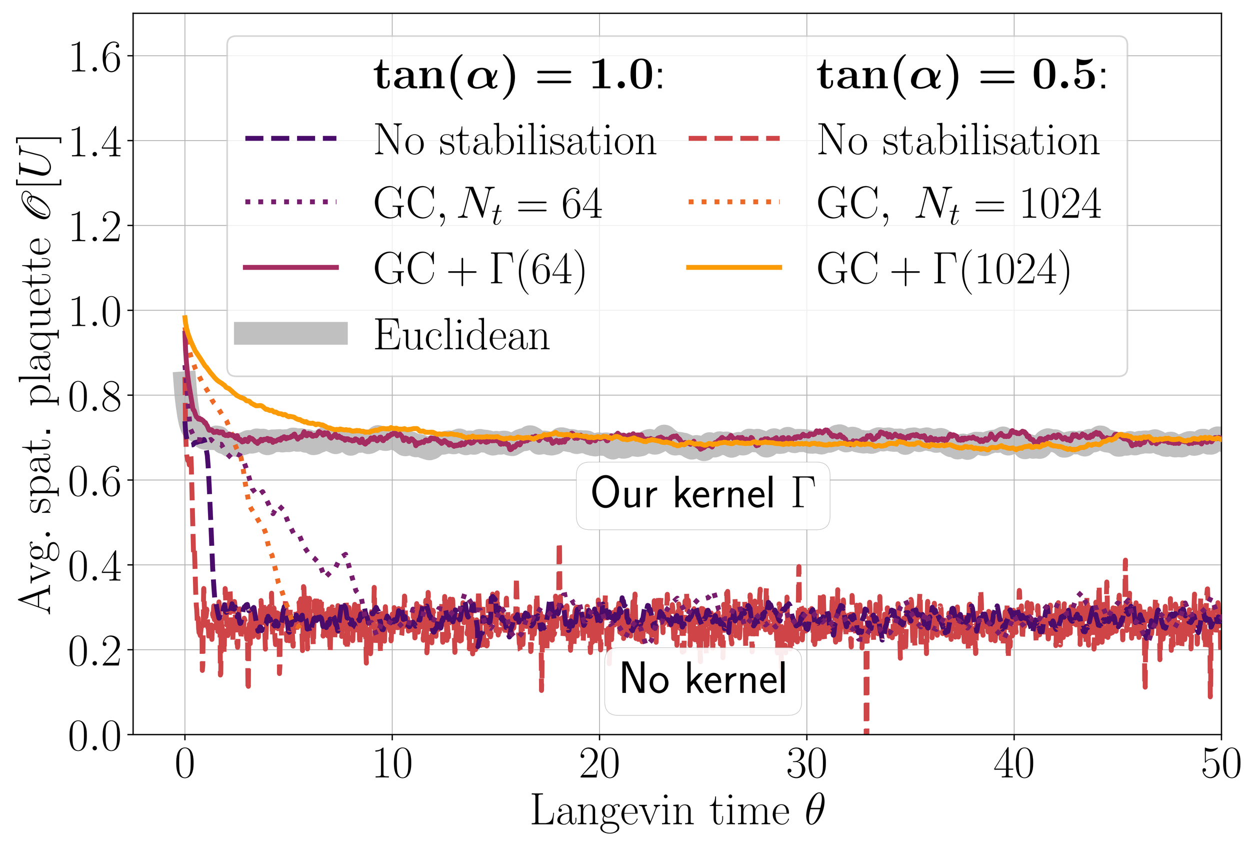

Figure 2: $\mathcal{O}$ for contour tilt angles α of Fig. 1 with AS and (i) no further stabilization, (ii) with GC and adjusted N_t, and (iii) with GC and our kernel $\Gamma(N_t)$. The gray curve shows the result on a Euclidean (thermal) time path.

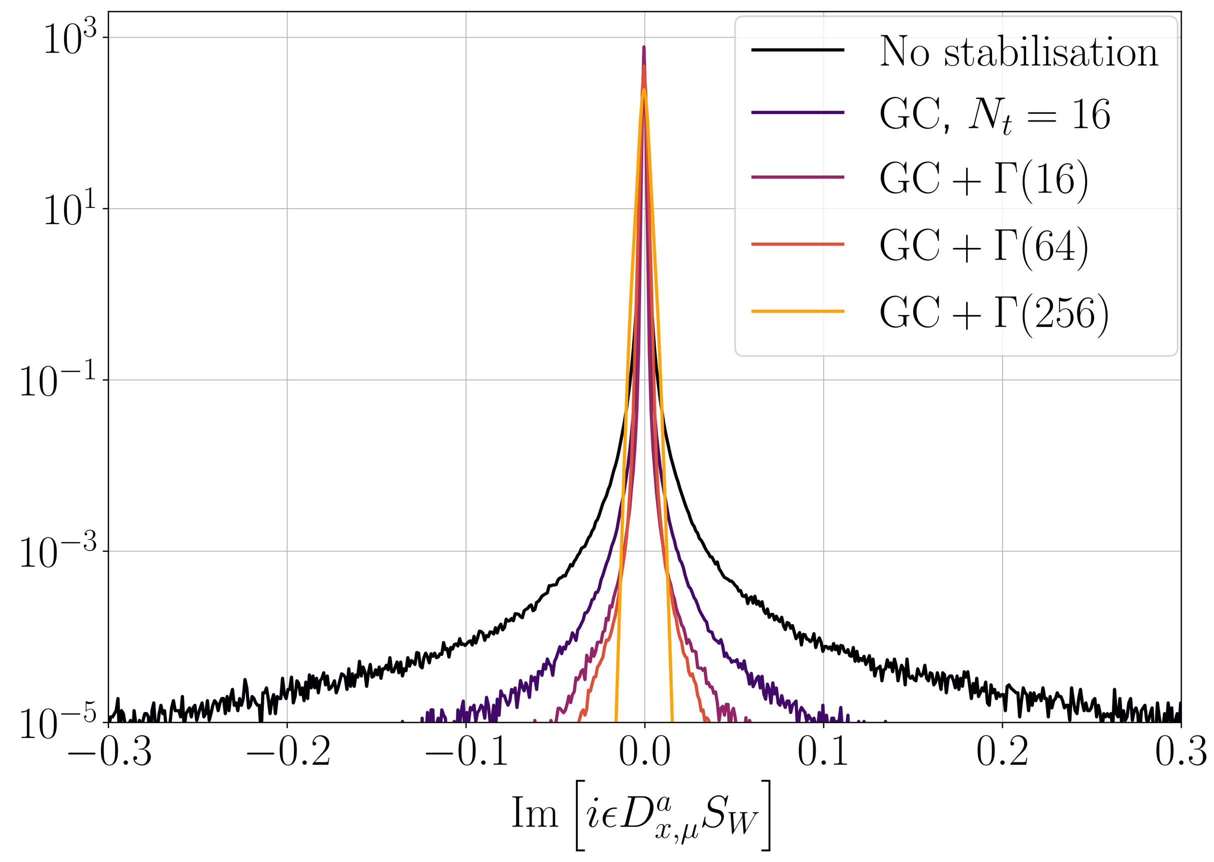

Figure 3: Normalized histogram of the non-unitary part of the drift term (i) without stabilization, (ii) with GC and adjusted N_t, and (iii) with GC and our kernel $\Gamma(N_t)$.

- ▹ Studies of SU(2) Yang-Mills theory in Ref. [3] on the tilted real-time path (yellow in Fig. 1) used the unkerneled CL equation (1) with $\Gamma_\mu = 1$. We **reproduce their results** as dashed lines in **Fig. 2** for the average spatial plaquette ($\mathcal{O} = \mathrm{ReTr}U_{ij}$).
- ▹ They quickly **converge to a wrong value** in **Fig. 2** although $\langle\mathcal{O}\rangle$ should be time-path independent and agree with the Euclidean result (gray) due to thermal time-translation invariance. Additional GC and increasing N_t for fixed $N_t a_t$ does not improve convergence (dotted).
- ▹ Exploiting the kernel freedom of CL, in Ref. [4] we introduce a **new anisotropic kernel** with $\Gamma_0 = |a_t|^2/a_s^2$ and $\Gamma_i = 1$ in Eq. (1), which we motivate using a new and unambiguous contour parameter formulation of the CL equation. In **Fig. 2** simulations with our kernel and the same N_t as before form a broad meta-stable θ-region that yields the correct thermal result after averaging over it (high precision).
- ▹ The improved convergence can be also seen in **Fig. 3** where we show the histogram of the imaginary part of the drift $DS_W = \frac{\delta S_W}{\delta\tilde{A}}$ for $\tan(\alpha) = 0.625$. Without our kernel, the stochastic process strays deep into the complex configuration space, leading to instabilities or wrong convergence. In contrast, $\Gamma(N_t)$ yields increasingly **localized distributions** with growing N_t, and thus correct CL processes.

5 Conclusion

- ▹ SU(2) gauge theory on complex time paths with CL requires additional stabilisation
- ▹ Our kernel Γ improves stability and leads to correct convergence

⇒ Extrapolation to Schwinger-Keldysh time contour, renormalization and scale setting

⇒ Potential application to transport coefficients, non-equilibrium dynamics of QCD

6 References

[1] G. Aarts, F. A. James, E. Seiler and I. O. Stamatescu, Phys. Lett. B **687**, 154-159 (2010).

[2] E. Seiler, D. Sexty and I. O. Stamatescu, Phys. Lett. B **723**, 213-216 (2013).

[3] J. Berges, S. Borsanyi, D. Sexty and I. O. Stamatescu, Phys. Rev. D **75**, 045007 (2007).

[4] K. Boguslavski, P. Hotzy and D. I. Müller, [arXiv:2212.08602].

Contact: **kirill.boguslavski@tuwien.ac.at**, **paul.hotzy@tuwien.ac.at**, **david.mueller@tuwien.ac.at** ITP, TU Wien

Hard Probes 2023 – Aschaffenburg, Germany